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Doctoral Thesis

SOME PROBLEMS OF FIXED POINT THEOREMS IN DIFFERENT SPACES

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A thesis submitted in fulfilment of the requirements

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Guru Ghasidas Vishwavidyalaya, Koni, Bilaspur

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By

Rashmi(M. Phil)

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Declaration

I do hereby declare that the work which is being presented in this thesis entitled "Some Problems of Fixed Point Theorems in Different Spaces" in the partial fulfillment of the requirement for the award of Doctor of philosophy in Mathematics submitted in the Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.).

The matter embodied in this thesis has not been submitted by me for the awarded of any other degree in any institution.

Ph.D. Student

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Certificate

This is to certify that Rashmi, student of Integrated M.Phil/Ph.D. Course in Mathematics, worked under my supervision on "Some Problems of Fixed Point Theorems in Different Spaces". She fulfills all the conditions for the award of the degree of Doctor of Philosophy in Mathematics from Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.). This work is the outcome of her studies during the academic session 2010 - 2014. This is her own work produced by consulting me, research papers, reviewed articles and books, etc. I forward this to the examiner for evaluation.

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Organization of the chapters

The thesis comprises of seven chapters. In chapter - I, we have given a brief history of Fixed Point Theory and its applications. Also, in this chapter we recall related several definitions, theorems.

In chapter - II, we have introduced the concept of tripled best proximity point in metric spaces. As an application to our main result, we prove some tripled best proximity point theorems in metric spaces which are the generalizations of recent results of Sintunavarat and Kumam([170]). Some examples and application in integral equations are given to support our results. Also our result improves the results of Babu and Alemayehu([60]) in metric spaces by giving shorter proof than of Babu and Alemayehu([60]).

In chapter - III, we have developed tripled coincidence point theorems in ordered cone metric spaces over a solid cone. Our results extend some coupled common fixed point theorems of Nashine, Kadelburg and Radenovic([68]).

Chapter - IV is divided into three sections:
In section - I, we have introduced the concept of compatible maps in Fuzzy metric spaces to establish triple coincidence point Theorem. Our result extends the work of Hu([172]) and we have supported our result by a suitable example.

In section - II, we have developed the notion of (f, g)-reciprocal continuity and proved a common fixed point theorem for a pair of sub-compatible maps by employing a generalized (φ, ψ)-weak contraction condition in fuzzy metric spaces. As an application of these results in Fuzzy Metric Spaces, we present a theorem that involves (φ, ψ)-weak cyclic contraction condition.

Section - III deals with a common fixed point theorem in Fuzzy metric spaces. Here we established the *Presič* type fixed point theorem([141]) and obtain the uniqueness of common fixed point for three maps in Fuzzy metric spaces. Also, we have deduced the main theorem of George([129]) as a corollary to our theorem under some conditions.

In chapter - V, we have proved some periodic point and fixed point theorems for a rational inequality in complex valued metric spaces. Also, we have given an application for our main theorem in complex valued metric spaces.

In chapter - VI, we focus on various tools of obtaining Fixed Point such as *various kinds of compatible maps* (for example: Compatible Maps of Type(A), Type(B), Type(C), Type(P), weakly compatible maps, etc.) for details of **tools** in Fixed Point Theorems we refer Murthy([124]). We have shown that these pairs satisfy a contractive condition of integral type in Menger Space using the above tools. Also, we improve the results of Branciari ([5]), Rhoades ([15]), Kumar, Chugh and Vats([153]). In addition, we have introduced the notion of ***Universal weakly compatible maps*** and proved a fixed point theorem for a weakly compatible pair of maps.

In chapter - VII, we have introduced a new notion "f-g reciprocal continuity in G-metric spaces" and proved common fixed point theorem for generalized (φ, ψ)-weak contraction condition. As an application of our result, we prove a theorem for (φ, ψ)-weak cyclic contraction condition.

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Chapter 1

INTRODUCTION

1.1

By fixed point, we shall mean a point which is invariant under any transformation. Let X be a non-empty set and let $T : X \to X$. If $T(x) = x$, then x is a fixed point of T. The importance of Fixed Point Theory lies in solving equations of the type $T(x) = 0$. For example, consider an equation $x^2 - 6x + 5 = 0$. In this equation we know that $x = 1$ and $x = 5$ are the roots of this equation. This equation can be written as $x = \frac{x^2+5}{6}$. If we write $T(x) = \frac{x^2+5}{6}$, then $x = 1$ and $x = 5$ are two fixed points of T. Thus, it is easy to see that the problem of finding the solution of an equation is the same as to find a fixed point for an equation of the form $T(x) = x$.

"Fixed point theory and applications", one of the most interesting and popular branch of Mathematics, comes under Functional Analysis; it is growing very rapidly in the form of developing theory and applications. Their applications are widely used in mathematical economy, mathematical physics, engineering and game theory, etc.

The Modern Fixed Point Theory originated as a consequence of proving the existence of solutions to a differential equation. In 1844, Cauchy([1]) was the first who gave an independent proof for the existence and uniqueness of the solution of the differential equation

$$\frac{dy}{dx} = T(x, y),$$

where T is continuously differential function. Later Lipschitz([133]) simplified Cauchy's proof in 1877, using a condition that is called "Lipschitz condition." In 1890, Peano ([51]) attempted a deeper result of Cauchy's theorem by supposing only the continuity of T. In 1930, Caccioppoli([128]) established a fixed point theorem by replacing the contraction property by convergence property. In 1910, Brouwer([95]) proved a fixed point theorem in which, if C is a unit ball in R^n and $T : C \to C$ be a continuous function, then T has a fixed point in C.

In 1927, Schauder([78]) extended the above result of Brouwer by assuming C as a compact convex set in his theorem. Again Schauder([79]) proved in 1930, that any compact convex nonempty subset of a normed linear space has the fixed point property. Tychonoff([10]) in 1935, proved that any compact convex nonempty subset of a locally convex space has the fixed point property. By fixed point property, we shall mean that every continuous mapping of a topological space X into itself has a fixed point.

Now we recall the classical Banach Fixed Point Theorem (1922) in metric spaces:

Theorem 1.1.1([134]) Let (X, d) be a complete metric space and T be a mapping from X into X such that

$$d(T(x), T(y)) \leq \alpha.d(x, y), \quad \text{where for all} \quad x,\ y \in X \text{ and } \alpha \in [0, 1),$$

if $\{x_n\}$ is a sequence in X such that $x_n = T^n(x_0)$, where $x_0 \in X$ be an arbitrary point of X. Then T has a unique fixed point in X.

These have been a number of generalizations of this result by Bryant([158]), Ciric([98] - [100]), Edelstein([102]), Sehgal ([159] - [160]), Jaggi and Das([41]), etc.

Edelstein([103]) generalized contraction principle due to Banach([134]) in a complete metric space called contractive mapping principle. Chu and Diaz([142]) and Sehgal([160]) extended this mapping.

Sehgal's result was further extended and generalized by Guseman([96]), Khazanchi([101]), Rhoades and Ray([20]), Iseki([84]) and Sharma([9]).

Rakotch([44]), Browder([47]), Boyd and Wong([42]) replaced Lipschitz constant by some real valued functions which are less than unity.

After 1960, it was quiet natural for the community to ask a question about fixed point that "*Does there exist a contractive mapping which does not force a mapping T to be continuous*". It was Kannan([131]) who answered this question affirmatively and proved a fixed point theorem for a mapping which does not satisfy the condition due to Banach([134]).

The contraction condition introduced by Kannan([131] is as follows:

there exists a number, $0 \leq \beta < \frac{1}{2}$ such that

$$d(T(x), T(y)) \leq \beta[d(x, T(x)) + d(y, T(y))] \quad \text{for all} \quad x, y \in X$$

After Kannan's([131]) contraction condition a variety of contractive conditions were introduced by many researchers. Some of them are Kannan([132]), Reich([150]), Fukshima([64]) , Bianchini([130]), Chatterjee([146]), Zamfirescu([156]), Hardy - Roger([50]), etc in metric and their related spaces. Although their contractive conditions are more general than of Banach([134]) and Kannan([131]), their corresponding proofs are basically same.

In 1976, Jungck ([52]) gave a generalization of the Banach's Fixed Point Theorem for a pair of commuting self maps and one of the function as a continuous map in a complete metric space (X, d). Perhaps Jungck is the first who studied these three conditions at a time i.e., Commuting pair of maps, continuous map and containment of ranges in the history of fixed point theory and applications.

Jungck's theorem inspired the Fixed Point Theorist as well as many applied discipline researchers to generalize the contraction and contractive type mappings in many ways (for eg. see Das and Naik [85], Jungck[52], Chang([143]), Fisher([16] - [19]), Prasad[34] and jungck[53]).

Various kinds of compatibility:

Two self maps f and g of a metric space (X, d) are said to be commuting if $fg(x) = gf(x)$ for all x in X.

Sessa([145]) generalized the concept of commuting mappings and introduced weakly commuting maps. Self - mappings f and g of a metric space (X, d) are called weakly commuting pair if

$$d(fg(x), gf(x)) \leq d(g(x), f(x))$$

for all x in X.

Obviously, commuting maps are weakly commuting but the converse is not true.

In 1986, Jungck([53]) introduced the concept of compatible maps in a metric space as a generalization of commuting maps and weakly commuting mappings. One can observe that every pair of weakly commuting maps is compatible but the converse is not true. Refer for examples([53] - [55], [144] etc.).

An equivalent definition of compatible maps is given by Jungck, Murthy and Cho by introducing the concept of compatible maps of type(A) and shown that compatible maps and compatible maps of type(A) are independent to each other. Refer for details and examples([56]).

Reader may find a review on such type of pair of maps having the property of non - commuting pair in metric spaces given by Murthy.([124]).

Tripled Best Proximity Point in metric spaces:

One of the most exciting result is the extension of Banach Fixed Point Theorem in complete metric space for non - self mapping as such that A to B does not necessarily have a fixed point where A and B are nonempty closed subsets of a complete metric space (X, d), with $A \cap B = \phi$.

A point x in A for which $d(x, T(x)) = d(A, B)$ is called a best proximity point of T, whenever a non - self mapping T has no fixed point, where $d(A, B) = \inf\{d(x, y)$:

$x \in A, y \in B\}$. A best proximity point represents an optimal solution to the equation $T(x) = x$. Since best proximity point reduces to a fixed point, if the underlying mapping is assumed to be self - mappings, the Best Proximity Point Theorems are natural generalizations of the Banach Fixed Point Theorem.

Example 1.1.2 Let $X = \mathbb{R}$ be endowed with the metric $d(x,y) =\mid x - y \mid$ and let $A = [2,4]$ and $B = [-4,-2]$. Clearly, $d(A,B) = 4$.

Define $T : A \to B$ such that $F(x) = -x$. Here we do not a have fixed point.

It is remarkable that Best Approximation Theorem is an approximate solution to the equation $T(x) = x$, when T has no solution. Indeed, the well-known best approximation theorem of Fan ([82]) asserts that if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B, and $T : A \to B$ is a continuous mapping, then there exists an element x satisfying the condition $d(x, T(x)) = \inf_{y\in A} d(T(x), y)$. Later, several researchers such as Reich [151], Prolla ([77], Sehgal and Singh [161] - [162], Vetrivel, Veeramani and Bhattacharyya [166], Wlodarczyk and Plebaniak ([88]-[90]), Eldred and Veeramani [2], Mongkolkeha and Kumam [29] and Sadiq Basha and Veeramani [138]-[140], Sadiq Basha [137]) obtained extensions of Fan's theorem in many directions.

In this context, we mildly modify the concept of tripled fixed point introduced by Brinde and Borcut([157]). We introduce the following notion of tripled best proximity point.

Definition 1.1.3 Let A and B be nonempty subsets of a metric space (X,d) and $F : A \times A \times A \to B$ be a mapping than an element $(x,y,z) \in A \times A \times A$ is called a tripled best proximity point of F, if

$$d(x, F(x,y,z)) = d(y, F(y,z,x)) = d(z, F(z,x,y)) = d(A,B).$$

where $d(A,B) = inf\{d(a,b) : a \in A, b \in B\}$.

Triplet coincidence point theorem in ordered cone metric spaces :

The concept of cone metric spaces was introduced by Huang and Zhang([97]) which is the generalization of a metric space. In this they have replaced $d : X \times X \to R$ by $d : X \times X \to E$, where E is the set of real Banach Space.

In 2004, the concept of partially ordered metric space(POMS) was introduced by Ran and Reurings ([20]). Guo and Lakshmikantham ([37]) studied the concept of coupled fixed points in POMS. Later, Bhaskar and Lakshmikantham ([154]) studied monotone property in POMS and supported this by providing an application to the existence of periodic boundary value problems. Recently, Karapinar([46]) proved coupled fixed point theorems for nonlinear contractions in ordered cone metric spaces over normal cones without regularity. He assumed the continuity and commutativity of both mappings in complete space. Shatanawi ([167]) proved coupled coincidence and coupled fixed point theorems in cone metric spaces which were not necessarily normal. Some results on this are due to Sabetghadam ([48]), Ding and Li ([66]), and Aydi, Samet and Vetro([67]).

According to Borcut and Berinde ([157]), we shall furnish the following definition(1.1.4 and 1.1.6):

Consider partial ordering on $X \times X \times X$ in the following manner:

for $(x, y, z), (u, v, w) \in X \times X \times X$, $(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w$.

Definition 1.1.4 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$ and $g : X \to X$. The mapping F is said to have mixed g - monotone property if F is monotone g - non-decreasing in x and z, is monotone g - non-increasing in y i.e., for any $x, y, z \in X$

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z) \tag{1.1.1}$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z) \tag{1.1.2}$$

$$z_1, z_2 \in X, g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1) \leq F(x, y, z_2) \tag{1.1.3}$$

hold.

Remark 1.1.5 g - monotone property on $X \times X$ is introduced by Lakshmikantham and Čirič([163]).

Definition 1.1.6 Let $(X, d, \preceq)$ be a nonempty ordered cone metric space and $F : X \times X \times X \to X$, $g : X \to X$. An element $(x, y, z) \in X \times X \times X$ is called:

(T_1) a tripled coincidence point of mappings F and g if $F(x, y, z) = g(x)$, $F(y, x, y) = g(y)$, $F(z, y, x) = g(z)$.

(T_2) a tripled fixed point of mapping F if $F(x, y, z) = g(x) = x$, $F(y, x, y) = g(y) = y$ and $F(z, y, x) = g(z) = z$.

Triplet coincidence point theorem in Fuzzy metric spaces:

Zhu and Xiao([174]) and Hu([172]) gave a coupled fixed point theorem for contractions in fuzzy metric spaces, and Fang([83]) proved some common fixed point theorems under ϕ-contractions for compatible and weakly compatible mappings on Menger probabilistic metric spaces.Very recently, the concept of tripled fixed point has been introduced by Berinde and Borcut ([157]). In their paper, some new tripled fixed point theorems are obtained using the mixed g-monotone mapping. Their results generalize and extend the results of Bhaskar and Lakshmikantham([154]).

Now we furnish here some definitions for our theorems:

Definition 1.1.7 An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of the map $F : X \times X \times X \to X$ and $g : X \to X$ if $F(x, y, z) = g(x) = x$, $F(y, x, z) = g(y) = y$ and $F(z, y, x) = g(z) = z$.

Definition 1.1.8 An element $(x, y, z) \in X \times X \times X$ is called a tripled coincidence point of the maps $F : X \times X \times X \to X$ and $g : X \to X$ if $F(x, y, z) = g(x)$, $F(y, x, z) = g(y)$ and $F(z, y, x) = g(z)$.

Presič type mapping and Common fixed point theorem in Fuzzy metric spaces:

In 1965, $Presi\check{c}$([141]) extended Banach's contraction mapping principle to operators defined on product spaces which follows:

Theorem 1.1.9 Let (X, d) be a metric space, k a positive integer, $X^k \equiv$ product of $X, k-$ times and $T : X^k \to X$ be a mapping satisfying the following condition:

$$d(T(x_1, x_2, \cdots, x_k), T(x_2, x_3, \cdots, x_{k+1})) \leq q_1.d(x_1, x_2)+q_2.d(x_2, x_3)+\cdots+q_k.d(x_k, x_{k+1})$$

where $x_1, x_2, \cdots, x_{k+1}$ are arbitrary elements in X and $q_1, q_2, \cdots, q_k$ are non-negative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then, there exists some $x \in X$ such that $x = T(x, x, \cdots, x)$. Moreover, if $x_1, x_2, \cdots, x_k$ are arbitrary points in X and for $n \in N$, $x_{n+k} = T(x_n, x_{n+1}, \cdots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $limx_n = T(limx_n, limx_n, \cdots, limx_n)$.

Note that for $k = 1$ the above theorem reduces to the well-known Banach Contraction Principle.

In this section, we have proved a common fixed point theorem of $Presi\check{c}$ type mapping in the setting of Fuzzy metric spaces which extends the results of George ([129]).

Common fixed point theorems in complex valued metric space:

This study has been already done by many mathematicians and we have another view point to revisit the study of complex valued metric spaces. The study of new discoveries in mathematics and their basic properties are always favorite topics of interest among the mathematical research community. In this context, the concept of 2-metric spaces was introduced initially by Gähler([135]) in his series of papers and he gave a thought on dimensions of ordinary metric spaces. Since the metric for a pair of points is non-negative real, (i.e. $[0, \infty)$). The concept of probabilistic metric spaces in which the probabilistic distance between two points is considered has given a new height and interest to the study to know more about stars in the universe. In a similar way, the study of fuzzy metric spaces was initially done by Grabiec([105]) and Erceg([104]) in which the degree of agreement and disagreement were considered.

So far, the study was done around the real numbers for e.g. Metric spaces, 2-Metric spaces, Normed linear spaces, Fuzzy metric spaces, Probabilistic metric spaces, etc. Let X be a non-empty set and let $d : X \times X \to R$, $||.|| : N \to R$, $d : X \times X \times X \to R$ and $M(x, y, t) : X \times X \times [0, \infty] \to [0, 1]$, $F : X \times X \longrightarrow [0, 1]$. It was quiet natural to ask "What happens if we replace R by some other sets which are not necessarily completely ordered sets like R?" and this was answered by a few researchers by introducing the cone metric spaces, the partially ordered metric spaces, the modular metric spaces and very recently, the complex valued metric spaces, respectively by Huang and Zhang([97]), Matthew ([136]), Chistyakov([165]), Azam, Fisher and Khan ([4]).

Here, first we shall introduce the concept of complex valued metric space by modifying the same introduced initially by Azam, Fisher and Khan([4]). As an application of these concepts, we shall construct a theorem and its application in integral equations.

Common fixed point theorems in Menger spaces:

In metric spaces $d(x, y)$ stands for distance between two points x and y. If the exact distance is not known, then the concept of probable distance comes. The concept of a Probabilistic Metric Space (in short PM-space) is used to deal with such situations. The notion of Probabilistic Metric Space (or Statistical Metric Space) was initially introduced by Menger([87]) in 1944, which is a generalization of Metric spaces. The idea in Probabilistic Metric space is associated with distribution functions with a pair of points, say (p, q), denoted by $F(p, q; t)$ where $t > 0$ and interpreted this function as the probability that distance between p and q is less than t. The study of PM-spaces was expanded rapidly with the pioneering works of Schweizer-Sklar([76], [70], [71], [72], [164], etc).

Here, we establish a fixed point theorem for integral type contraction condition in the probabilistic metric space for a pair of compatible maps of type (A) (or compatible of type (P) or weak compatible of type (A)) maps.

Generalized (φ, ψ)-Weak Contractions in G - metric spaces:

Now we recall the following from the History of Fixed Point Theory and Applications: Boyd and Wong in 1969 ([42]) introduced such type of functions to generalized Contraction Condition due to Banach:

Theorem 1.1.10 Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$. If $T : X \to X$ satisfies

$$d(T(x), T(y)) \leq \psi(d(x, y)) \quad \text{for all} \quad x, y \in X,$$

then it has a unique fixed point $x \in X$ and $\{T^n(x)\}$ converges to x for all $x \in X$.

Later, Chang([143]) generalizing and unifying fixed point theorems of Jungck([52]), Das and Naik([85]), etc.

Theorem 1.1.11 Let (X, d) be a complete metric space, a self - mapping f on X such that for some positive integer m, f^m is continuous. Let $\{g_i\}_{i=1}^{\infty} : f^{m-1}(X) \to X$ be a sequence of mappings such that

$$g_i(f^{m-1}(X)) \subset f^m(X), \qquad i = 1, 2, \cdots,$$

and suppose g_i commutes with $f, i = 1, 2, \cdots$. Further, suppose there exists a sequence of positive integers $\{m_i\}_{i=1}^{\infty}$ such that for any positive integers i, j and any $x, y \in f^{m-1}(X)$ the following holds

$$d(g_i^{m_i}(x), g_j^{m_j}(y))$$

$$\leq A(max\{d(f(x), f(y)), d(f(x), g_i^{m_i}(x)), d(f(y), g_j^{m_j}(y)),$$

$$d(f(y), g_i^{m_i}(x)), d(f(x), g_j^{m_j}(y))\}),$$

where the function A satisfies the following conditions $(A1), (A2)$ or $(A1), (A3)$:

$(A1) A : [0, \infty) \to [0, \infty)$ is nondecreasing and right continuous.

$(A2)$ For any real number $q \in [0, \infty)$ there exists a suitable real number $t(q) \in [0, \infty)$ such that

(a) $t(q)$ is the upper bound of the set

$\{t \in [0, \infty) : t \leq q + A(t)\},$

(b) $\lim_{n\to\infty} A^n(t(q)) = 0.$

$(A3)$ For every $t > 0, A(t) < t$ and

$$\lim_{t\to\infty}(t - A(t)) = \infty.$$

Then f and $g_i,\ i = 1, 2, \cdots$ have a unique common fixed point $f(y^*)$, where $y^* \in X$ is the limit of the sequence $\{y_n\}_{n=1}^{\infty}$ defined as $y_n = g_n^{m_n}(x_n) = f(x_{n+1}), n = 1, 2, \cdots$.

Alber and Guerre-Delabriere ([173]) suggested a generalization of the Banach contraction mapping principle by introducing the concept of weak contraction in Hilbert spaces. Later, Rhoades ([14]) realized that this result of Hilbert spaces is valid in complete metric spaces also.

Rhoades ([14]) proved the following theorem and the contractive condition introduced called as weak contraction:

Definition 1.1.12 Let X be a metric space. A mapping $T : X \to X$ is called contractive if and only if

$$d(T(x), T(y)) \leq d(x, y) - \varphi(d(x, y)),$$

$\forall \quad x, y \in X$, where φ is an altering distance function.

In 1984, Khan, Swaleh and Sessa ([116]) have introduced the concept of an altering distance function in metric spaces in the following manner:

Definition 1.1.13 A function $\varphi : [0, \infty) \to [0, \infty)$ is an altering distance function if $\varphi(t)$ is monotone non-decreasing, continuous and $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 1.1.14 [[14], Theorem 2] Let (X, d) be a complete metric space. If $T : X \to X$ is a weakly contractive mapping, then T has a unique fixed point.

Dutta and Choudhury([122]) further generalized the weak contraction condition of Rhoades ([14] in particular Theorem 2).

Theorem 1.1.15 Let (X, d) be a complete metric space and $T : X \to X$ satisfying

$$\psi(d(T(x), T(y)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

$\forall \quad x, y \in X$ where ψ and φ are altering distance function. Then T has a unique fixed point.

Many results are available using (φ, ψ)-Weak Contractions in metric spaces as well as in its related spaces. One can see some of them in (Zhang and Song ([127]), Dorić ([35])). Here, we deal with such type of contractions having control functions. In fact, all contractions are the generalizations of Banach by including the distance function within the real valued function. We have proved some fixed point theorems in G-metric space by using (ϕ, ψ) - contraction condition.

Chapter 2

TRIPLED BEST PROXIMITY POINTS AND COMMON FIXED POINTS IN METRIC SPACES

2.1

Present chapter has been divided into two section. In this chapter, we shall prove two types of theorems in metric spaces.

1. In section - I, we shall prove tripled best proximity point theorems supported by examples and at the end we shall give an application in integral equations of our result.

2. In section - II, we improve the results of Babu and Alemayehu ([60]) in metric spaces by giving shorter proof than of Babu and Alemayehu ([60]).

Throughout this section $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{R}$ denotes the set of all real numbers and $d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$, where $A \cap B = \phi$.

2.2 Section I

We need the following important definitions for proving our main theorem.

Definition 2.2.1[155] Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC if the following holds:

if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = d(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = d(A, B)$, then $\lim_{n\to\infty} d(x_n, z_n) = 0$.

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Definition 2.2.2[170] Let A and B be nonempty subsets of a metric space (X,d). The ordered pair (A,B) satisfies the property UC^*, if (A,B) has the property UC and the following condition hold:
if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B satisfying:

(i) $d(z_n, y_n) \to d(A,B)$ as $n \to \infty$,

(ii) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, y_n) \leq d(A,B) + \epsilon$, for all $m > n \geq N$,

then, for every $\epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $d(x_m, z_n) \leq d(A,B) + \epsilon$ for all $m > n \geq N_1$.

Definition 2.2.3[157] Let A be a nonempty subset of a set POMS X and $F : A \times A \times A \to A$ be a mapping. An element $(x,y,z) \in A \times A \times A$ is called a *tripled fixed point* of F if

$$x = F(x,y,z),\ y = F(y,z,y) \text{ and } z = F(z,x,y).$$

In view of the above concepts, now we shall introduce the notion of tripled best proximity point in metric spaces by modifying the notion of the same introduced by Brinde and Borcut([157]) in partially ordered metric spaces.

Definition 2.2.4 Let A and B be nonempty subsets of a metric space (X,d) and $F : A \times A \times A \to B$ be a mapping. An element $(x,y,z) \in A \times A \times A$ is said to be *tripled best proximity point* of F if

$$d(x, F(x,y,z)) = d(y, F(y,z,x)) = d(z, F(z,x,y)) = d(A,B).$$

It is easy to see that, if $A = B$ in above definition, then a tripled best proximity point reduces to a tripled fixed point.

Now, we are going to introduce the notion of a *cyclic contraction* for a pair of mappings in metric spaces by modifying slightly the notion which was initially introduced by Eldred and Veeramani([2]).

Definition 2.2.5 Let A and B be nonempty subsets of a metric space (X, d). The maps $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ is said to satisfy *cyclic contraction* for some $\alpha \in [0, 1)$;

$$d(F(x, y, z), G(u, v, w)) \leq \frac{\alpha}{3}[d(x, u) + d(y, v) + d(z, w)] + (1 - \alpha)d(A, B),$$

for all $(x, y, z) \in A \times A \times A$ and $(u, v, w) \in B \times B \times B$.

Remark 2.2.6 If (F, G) is a cyclic contraction, then (G, F) is also a cyclic contraction.

Example 2.2.7 Let $X = \mathbb{R}$ endowed with the metric $d(x, y) =\mid x-y \mid$ and let $A = [2, 4]$ and $B = [-4, -2]$. Clearly, $d(A, B) = 4$.
Define $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ by

$$F(x, y, z) = \frac{-x - y - z - 6}{6} \text{ and } G(u, v, w) = \frac{-u - v - w + 6}{6}.$$

Fix $\alpha = \frac{1}{2}$ for arbitrary $(x, y, z) \in A \times A \times A$ and $(u, v, w) \in B \times B \times B$, we get

$$d(F(x, y, z), G(u, v, w)) =\mid \tfrac{-x-y-z-6}{6} - \tfrac{-u-v-w+6}{6} \mid$$

$$\leq \tfrac{|x-u|+|y-v|+|z-w|}{6} + 2$$

$$= \tfrac{\alpha}{3}[d(x, u) + d(y, v) + d(z, w)] + (1 - \alpha)d(A, B).$$

This implies that (F, G) is a cyclic contraction pair with $\alpha = \frac{1}{2}$.

Example 2.2.8 Let $X = \mathbb{R}^2$ endowed with the metric given by

$$d((x, y, z), (u, v, w)) = \max\{| x - u |, | y - v |, | z - w |\}$$

and let $A = \{(x, 0) : 0 \leq x \leq 1\}$ and $B = \{(x, 1) : 0 \leq x \leq 1\}$. Clearly, $d(A, B) = 1$. Define $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ by

$$F((x, 0), (y, 0), (z, 0)) = (\frac{x + y + z}{3}, 1) \text{ and } G((u, 1), (v, 1), (w, 1)) = (\frac{u + v + w}{3}, 0).$$

For arbitrary $x, y, z, u, v, w \in [0, 1]$, we get

$$\begin{aligned} d(F((x, 0), (y, 0), (z, 0)), G((u, 1), (v, 1), (w, 1))) &= d((\tfrac{x+y+z}{3}, 1), (\tfrac{u+v+w}{3}, 0)) \\ &= 1. \end{aligned}$$

For all $\alpha > 0$, we have

$$\begin{aligned} &\tfrac{\alpha}{3}[d((x, 0), (u, 1)), d((y, 0), (v, 1)), d((z, 0), (w, 1))] + (1 - \alpha)d(A, B) \\ &= \tfrac{\alpha}{3}[\max\{| x - u |, 1\} + \max\{| y - v |, 1\} \\ &\quad + \max\{| z - w |, 1\}] + (1 - \alpha)d(A, B) \\ &= \tfrac{\alpha}{3} \times 3 + (1 - \alpha) \\ &= 1. \end{aligned}$$

This implies that (F, G) is a cyclic contraction.

Before presenting our main Theorem, we need to prove the following lemmas:

Lemma 2.2.9 Let A and B be nonempty subsets of a metric space (X, d). A pair of maps F and G is satisfying cyclic contraction condition, if

$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}), y_{2n+1} = F(y_{2n}, z_{2n}, x_{2n}), z_{2n+1} = F(z_{2n}, x_{2n}, y_{2n}),$

for all $n \in \mathbb{N} \cup \{0\}$,

and

$x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}), y_{2n+2} = G(y_{2n+1}, z_{2n+1}, x_{2n+1}), z_{2n+2} = G(z_{2n+1}, x_{2n+1}, y_{2n+1})$

for all $n \in \mathbb{N} \cup \{0\}$,

then

$$
\begin{aligned}
\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = d(A, B), &\quad \lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) = d(A, B),\\
\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = d(A, B), &\quad \lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}) = d(A, B),\\
\lim_{n\to\infty} d(z_{2n}, z_{2n+1}) = d(A, B), &\quad \lim_{n\to\infty} d(z_{2n+1}, z_{2n+2}) = d(A, B).
\end{aligned}
$$

Proof. For each $n \in \mathbb{N} \cup \{0\}$, we have

$$
\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(x_{2n}, F(x_{2n}, y_{2n}, z_{2n}))\\
&= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), G(y_{2n-1}, z_{2n-1}, x_{2n-1}),\\
&\quad G(z_{2n-1}, x_{2n-1}, y_{2n-1})))\\
&\leq \tfrac{\alpha}{3}[d(x_{2n-1}, G(x_{2n-1}, y_{2n-1}, z_{2n-1})) + d(y_{2n-1}, G(y_{2n-1}, z_{2n-1}, x_{2n-1}))\\
&\quad + d(z_{2n-1}, G(z_{2n-1}, x_{2n-1}, y_{2n-1}))] + (1-\alpha)d(A, B)\\
&= \tfrac{\alpha}{3}[d(F(x_{2n-2}, y_{2n-2}, z_{2n-2}), G(F(x_{2n-2}, y_{2n-2}, z_{2n-2}), F(y_{2n-2}, z_{2n-2}, x_{2n-2}),\\
&\quad F(z_{2n-2}, x_{2n-2}, y_{2n-2})) + d(F(y_{2n-2}, z_{2n-2}, x_{2n-2}), G(F(y_{2n-2}, z_{2n-2}, x_{2n-2}),\\
&\quad F(z_{2n-2}, x_{2n-2}, y_{2n-2}), F(x_{2n-2}, y_{2n-2}, z_{2n-2})) + d(F(z_{2n-2}, x_{2n-2}, y_{2n-2}),\\
&\quad G(F(z_{2n-2}, x_{2n-2}, y_{2n-2}), F(x_{2n-2}, y_{2n-2}, z_{2n-2}), F(y_{2n-2}, z_{2n-2}, x_{2n-2}))]\\
&\quad + (1-\alpha)d(A, B)\\
&\leq \tfrac{\alpha}{3}\{\tfrac{\alpha}{3}[d(x_{2n-2}, F(x_{2n-2}, y_{2n-2}, z_{2n-2})) + d(y_{2n-2}, F(y_{2n-2}, z_{2n-2}, x_{2n-2}))
\end{aligned}
$$

$+d(z_{2n-2}, F(z_{2n-2}, x_{2n-2}, y_{2n-2})) + (1-\alpha)d(A,B)]$

$+\frac{\alpha}{3}[d(y_{2n-2}, F(y_{2n-2}, z_{2n-2}, x_{2n-2})) + d(z_{2n-2}, F(z_{2n-2}, x_{2n-2}, y_{2n-2}))$

$+d(x_{2n-2}, F(x_{2n-2}, y_{2n-2}, z_{2n-2})) + (1-\alpha)d(A,B)]$

$+\frac{\alpha}{3}[d(z_{2n-2}, F(z_{2n-2}, x_{2n-2}, y_{2n-2})) + d(x_{2n-2}, F(x_{2n-2}, y_{2n-2}, z_{2n-2}))$

$+d(y_{2n-2}, F(y_{2n-2}, z_{2n-2}, x_{2n-2})) + (1-\alpha)d(A,B)]$

$+(1-\alpha)d(A,B)\}$

$= \frac{\alpha^2}{3}[d(x_{2n-2}, F(x_{2n-2}, y_{2n-2}, z_{2n-2})) + d(y_{2n-2}, F(y_{2n-2}, z_{2n-2}, x_{2n-2}))$

$+d(z_{2n-2}, F(z_{2n-2}, x_{2n-2}, y_{2n-2}))] + (1-\alpha^2)d(A,B).$

By induction, we have

$$d(x_{2n}, x_{2n+1}) \leq \frac{\alpha^{2n}}{3}[d(x_0, F(x_0, y_0, z_0)) + d(y_0, F(y_0, z_0, x_0)) + d(z_0, F(z_0, x_0, y_0))]$$

$$+(1-\alpha^{2n})d(A,B),$$

and on taking $n \to \infty$, we get $d(x_{2n}, x_{2n+1}) \to d(A,B)$.

Arguing as above for each $n \in \mathbb{N} \cup \{0\}$, we have

$d(x_{2n+1}, x_{2n+2}) = d(x_{2n+1}, G(x_{2n+1}, y_{2n+1}, z_{2n+1}))$

$= d(F(x_{2n}, y_{2n}, z_{2n}), G(F(x_{2n}, y_{2n}, z_{2n}), F(y_{2n}, z_{2n}, x_{2n}), G(z_{2n}, x_{2n}, y_{2n})))$

$\leq \frac{\alpha}{3}[d(x_{2n}, F(x_{2n}, y_{2n}, z_{2n}))+d(y_{2n}, F(y_{2n}, z_{2n}, x_{2n}))+d(z_{2n}, F(z_{2n}, x_{2n}, y_{2n}))]$

$+(1-\alpha)d(A,B)$

$= \frac{\alpha}{3}[d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), G(y_{2n-1}, z_{2n-1}, x_{2n-1}),$

$G(z_{2n-1}, x_{2n-1}, y_{2n-1})) + d(G(y_{2n-1}, z_{2n-1}, x_{2n-1}), F(G(y_{2n-1}, z_{2n-1}, x_{2n-1}),$

$$G(z_{2n-1}, x_{2n-1}, y_{2n-1}), G(x_{2n-1}, y_{2n-1}, z_{2n-1}))] + d(G(z_{2n-1}, x_{2n-1}, y_{2n-1}),$$

$$F(G(z_{2n-1}, x_{2n-1}, y_{2n-1}), G(x_{2n-1}, y_{2n-1}, z_{2n-1}), G(y_{2n-1}, z_{2n-1}, x_{2n-1}))$$

$$+(1-\alpha)d(A, B)$$

$$\leq \tfrac{\alpha}{3}\{\tfrac{\alpha}{3}[d(x_{2n-1}, G(x_{2n-1}, y_{2n-1}, z_{2n-1})) + d(y_{2n-2}, G(y_{2n-1}, z_{2n-1}, x_{2n-1}))$$

$$+d(z_{2n-1}, F(z_{2n-1}, x_{2n-1}, y_{2n-1})) + (1-\alpha)d(A, B)]$$

$$+\tfrac{\alpha}{3}[d(y_{2n-1}, G(y_{2n-1}, z_{2n-1}, x_{2n-1})) + d(z_{2n-1}, G(z_{2n-1}, x_{2n-1}, y_{2n-1}))$$

$$+d(x_{2n-1}, G(x_{2n-1}, y_{2n-1}, z_{2n-1})) + (1-\alpha)d(A, B)]$$

$$+\tfrac{\alpha}{3}[d(z_{2n-1}, G(z_{2n-1}, x_{2n-1}, y_{2n-1})) + d(x_{2n-1}, G(x_{2n-1}, y_{2n-1}, z_{2n-1}))$$

$$+d(y_{2n-1}, G(y_{2n-1}, z_{2n-1}, x_{2n-1})) + (1-\alpha)d(A, B)] + (1-\alpha)d(A, B)\}$$

$$= \tfrac{\alpha^2}{3}[d(x_{2n-1}, G(x_{2n-1}, y_{2n-1}, z_{2n-1})) + d(y_{2n-1}, G(y_{2n-1}, z_{2n-1}, x_{2n-1}))$$

$$+d(z_{2n-1}, G(z_{2n-1}, x_{2n-1}, y_{2n-1}))] + (1-\alpha^2)d(A, B).$$

By induction, we get

$$d(x_{2n+1}, x_{2n+2}) \leq \tfrac{\alpha^{2n}}{3}[d(x_1, F(x_1, y_1, z_1)) + d(y_1, F(y_1, z_1, x_1)) + d(z_1, F(z_1, x_1, y_1))]$$

$$+(1-\alpha^{2n})d(A, B),$$

and on taking $n \to \infty$, we obtain $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$.

Similar arguing for $\{y_{2n}\}, \{y_{2n+1}\}, \{z_{2n}\}$ and $\{z_{2n+1}\}$, we will have $d(y_{2n}, y_{2n+1}) \to d(A, B)$, $d(y_{2n+1}, y_{2n+2}) \to d(A, B)$, $d(z_{2n}, z_{2n+1}) \to d(A, B)$ and $d(z_{2n+1}, z_{2n+2}) \to d(A, B)$ respectively.

Lemma 2.2.10 Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) and (B, A) satisfy the property UC and a pair of maps F and G satisfying cyclic contraction condition, if

$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}), y_{2n+1} = F(y_{2n}, z_{2n}, x_{2n}), z_{2n+1} = F(z_{2n}, x_{2n}, y_{2n}),$

for all $n \in \mathbb{N} \cup \{0\}$,

and

$x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}), y_{2n+2} = G(y_{2n+1}, z_{2n+1}, x_{2n+1}), z_{2n+2} = G(z_{2n+1}, x_{2n+1}, y_{2n+1}),$

for all $n \in \mathbb{N} \cup \{0\}$,

then for $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$,

$$\frac{1}{3}[d(x_{2m}, x_{2n+1}) + d(y_{2m}, y_{2n+1}) + d(z_{2m}, z_{2n+1})] < d(A, B) + \epsilon. \tag{2.2.1}$$

Proof. By Lemma 2.2.9, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$ as $n \to \infty$. Since (A, B) satisfies the property UC, we get $d(x_{2n}, x_{2n+2}) \to 0$ as $n \to \infty$. Similarly $d(y_{2n}, y_{2n+2}) \to 0$ and $d(z_{2n}, z_{2n+2}) \to 0$ as $n \to \infty$. Since (B, A) satisfies the property UC, we get $d(x_{2n+1}, x_{2n+3}) \to 0, d(y_{2n+1}, y_{2n+3})$ and $d(z_{2n+1}, z_{2n+3}) \to 0$ as $n \to \infty$.

If possible suppose that (2.2.1) does not hold. Then there exists $\epsilon' > 0$ such that for all $k \in \mathbb{N}$, there exist $m_k > n_k \geq k$ satisfying

$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \geq d(A, B) + \epsilon'$

and

$\frac{1}{3}[d(x_{2m_k-2}, x_{2n_k+1}) + d(y_{2m_k-2}, y_{2n_k+1}) + d(z_{2m_k-2}, z_{2n_k+1})] < d(A, B) + \epsilon'.$

Therefore, we get

$$\begin{aligned}
d(A, B) + \epsilon' &\leq \tfrac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
&\leq \tfrac{1}{3}[d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k-2}) \\
&\quad + d(y_{2m_k-2}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k-2}) + d(z_{2m_k-2}, z_{2n_k+1})] \\
&< \tfrac{1}{3}[d(x_{2m_k}, x_{2m_k-2}) + d(y_{2m_k}, y_{2m_k-2}) + d(z_{2m_k}, z_{2m_k-2})] + d(A, B) + \epsilon'.
\end{aligned}$$

Letting $k \to \infty$, we obtain

$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \to d(A, B) + \epsilon'$.

By using the triangle inequality we get

$$
\begin{aligned}
&\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
&\leq \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2m_k+3}) + d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) \\
&+ d(y_{2m_k+2}, y_{2m_k+3}) + d(y_{2m_k+3}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+2}, z_{2m_k+3}) \\
&+ d(z_{2m_k+3}, z_{2n_k+1})] \\
&= \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}), F(x_{2m_k+2}, y_{2m_k+2}, z_{2m_k+2})) \\
&+ d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) + d(G(y_{2m_k+1}, z_{2m_k+1}, x_{2m_k+1}), \\
&F(y_{2m_k+2}, z_{2m_k+2}, x_{2m_k+2})) + d(y_{2m_k+3}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k+2}) \\
&+ d(G(z_{2m_k+1}, x_{2m_k+1}, y_{2m_k+1}), F(z_{2m_k+2}, x_{2m_k+2}, y_{2m_k+2})) + d(z_{2m_k+3}, z_{2n_k+1})] \\
&\leq \frac{1}{3}\{d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{3}[d(x_{2m_k+1}, x_{2m_k+2}) + d(y_{2m_k+1}, y_{2m_k+2}) + d(z_{2m_k+1}, z_{2m_k+2})] \\
&+ (1-\alpha)d(A, B) + d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) \\
&+ \frac{\alpha}{3}[d(y_{2m_k+1}, y_{2m_k+2}) + d(z_{2m_k+1}, z_{2m_k+2}) + d(x_{2m_k+1}, x_{2m_k+2})] \\
&+ (1-\alpha)d(A, B) + d(y_{2m_k+3}, y_{2m_k+1}) + d(z_{2m_k}, z_{2m_k+2}) + \frac{\alpha}{3}[d(z_{2m_k+1}, z_{2m_k+2}) \\
&+ d(x_{2m_k+1}, x_{2m_k+2}) + d(y_{2m_k+1}, y_{2m_k+2})] + (1-\alpha)d(A, B) + d(z_{2m_k+3}, z_{2n_k+1})\} \\
&= \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) \\
&+ d(y_{2m_k+3}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+3}, z_{2n_k+1})] \\
&+ \frac{\alpha}{3}[d(x_{2m_k+1}, x_{2m_k+2}) + d(y_{2m_k+1}, y_{2m_k+2}) + d(z_{2m_k+1}, z_{2m_k+2})] + (1-\alpha)d(A, B)
\end{aligned}
$$

$$= \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) + d(y_{2m_k+3}, y_{2n_k+1})$$

$$+ d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+3}, z_{2n_k+1})]$$

$$+ \frac{\alpha}{3}[d(F(x_{2m_k}, y_{2m_k}, z_{2m_k}), G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}))$$

$$+ d(F(y_{2m_k}, z_{2m_k}, x_{2m_k}), G(y_{2m_k+1}, z_{2m_k+1}, x_{2m_k+1}))$$

$$+ d(F(z_{2m_k}, x_{2m_k}, y_{2m_k}), G(z_{2m_k+1}, x_{2m_k+1}, y_{2m_k+1}))] + (1-\alpha)d(A, B)$$

$$\leq \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2})$$

$$+ d(y_{2m_k+3}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+3}, z_{2n_k+1})]$$

$$+ \frac{\alpha}{3}\{\frac{\alpha}{3}[d(x_{2m_k}, x_{2m_k+1}) + d(y_{2m_k}, y_{2m_k+1}) + d(z_{2m_k}, z_{2m_k+1})] + (1-\alpha)d(A, B)$$

$$+ \frac{\alpha}{3}[d(y_{2m_k}, y_{2m_k+1}) + d(z_{2m_k}, z_{2m_k+1}) + d(x_{2m_k}, x_{2m_k+1})] + (1-\alpha)d(A, B)$$

$$+ \frac{\alpha}{3}[d(z_{2m_k}, z_{2m_k+1}) + d(x_{2m_k}, x_{2m_k+1}) + d(y_{2m_k}, y_{2m_k+1})] + (1-\alpha)d(A, B)\}$$

$$+ (1-\alpha)d(A, B)$$

$$= \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) + d(y_{2m_k+3}, y_{2n_k+1})$$

$$+ d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+3}, z_{2n_k+1})] + \frac{\alpha^2}{3}[d(x_{2m_k}, x_{2m_k+1}) + d(y_{2m_k}, y_{2m_k+1})$$

$$+ d(z_{2m_k}, z_{2m_k+1})] + (1-\alpha^2)d(A, B).$$

By taking $k \to \infty$, we get

$$d(A, B) + \epsilon' \leq \alpha^2[d(A, B) + \epsilon'] + (1-\alpha^2)d(A, B)$$

$$= d(A, B) + \alpha^2\epsilon',$$

which is a contradiction and so, we conclude that (2.2.1) holds true.

Lemma 2.2.11 Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) and (B, A) satisfying the property UC^* and a pair of maps F and G satisfying cyclic contraction condition, if

$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}), y_{2n+1} = F(y_{2n}, z_{2n}, x_{2n}), z_{2n+1} = F(z_{2n}, x_{2n}, y_{2n}),$

for all $n \in \mathbb{N} \cup \{0\}$,

and

$x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}), y_{2n+2} = G(y_{2n+1}, z_{2n+1}, x_{2n+1}), z_{2n+2} = G(z_{2n+1}, x_{2n+1}, y_{2n+1}),$

for all $n \in \mathbb{N} \cup \{0\}$,

then $\{x_{2n}\}, \{x_{2n+1}\}, \{y_{2n}\}, \{y_{2n+1}\}, \{z_{2n}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences.

Proof. By Lemma 2.2.9, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$ as $n \to \infty$. Since (A, B) satisfies the property UC*, we get $d(x_{2n}, x_{2n+2}) \to 0$ as $n \to \infty$. Also, since (B, A) satisfies the property UC*, we have $d(x_{2n+1}, x_{2n+3}) \to 0$ as $n \to \infty$. We now show that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_{2m}, x_{2n+1}) \leq d(A, B) + \epsilon, \tag{2.2.2}$$

for all $m > n \geq N$.

Suppose that (2.2.2) does not hold, then there is $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exist $m_k > n_k \geq k$ with

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon. \tag{2.2.3}$$

Now we have

$$\begin{aligned} d(A, B) + \epsilon &< d(x_{2m_k}, x_{2n_k+1}) \\ &\leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k+1}) \\ &\leq d(A, B) + \epsilon + d(x_{2n_k-1}, x_{2n_k+1}). \end{aligned}$$

Taking $k \to \infty$, we get $d(x_{2m_k}, x_{2n_k+1}) \to d(A, B) + \epsilon$.

Next, by Lemma 2.2.10, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] < d(A, B) + \epsilon,$$

for all $m > n \geq N$. By using the triangle inequality we get

$$\begin{aligned}
d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&= d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}), F(x_{2n_k+2}, y_{2n_k+2}, z_{2n_k+2})) \\
&\quad + d(x_{2n_k+3}, x_{2n_k+1}) \\
&\leq d(x_{2m_k}, x_{2m_k+2}) + \tfrac{\alpha}{3}[d(x_{2m_k+1}, x_{2n_k+2}) + d(y_{2m_k+1}, y_{2n_k+2}) \\
&\quad + d(z_{2m_k+1}, z_{2n_k+2})] + d(x_{2n_k+3}, x_{2n_k+1}) + (1-\alpha)d(A, B) \\
&= \tfrac{\alpha}{3}[d(F(x_{2m_k}, y_{2m_k}, z_{2m_k}), G(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1})) \\
&\quad + d(F(y_{2m_k}, z_{2m_k}, x_{2m_k}), G(y_{2n_k+1}, z_{2n_k+1}, x_{2n_k+1})) \\
&\quad + d(F(z_{2m_k}, x_{2m_k}, y_{2m_k}), G(z_{2n_k+1}, x_{2n_k+1}, y_{2n_k+1}))] \\
&\quad + (1-\alpha)d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&\leq \tfrac{\alpha}{3}\{\tfrac{\alpha}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
&\quad + (1-\alpha)d(A, B) + \tfrac{\alpha}{3}[d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1})] \\
&\quad + (1-\alpha)d(A, B) + \tfrac{\alpha}{3}[d(z_{2m_k}, z_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1})] \\
&\quad + (1-\alpha)d(A, B)\} + (1-\alpha)d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
&= \tfrac{\alpha^2}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})]
\end{aligned}$$

$$+(1-\alpha^2)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})$$

$$< \alpha^2(d(A,B)+\epsilon)+(1-\alpha^2)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})$$

$$= d(A,B)+\alpha^2\epsilon+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1}).$$

Taking $k \to \infty$, we get

$$d(A,B)+\epsilon \leq d(A,B)+\alpha^2\epsilon,$$

which is a contradiction and hence, condition (2.2.2) holds true. Since (2.2.2) holds and $d(x_{2n},x_{2n+1}) \to d(A,B)$, by using property UC* of (A,B), we deduce that $\{x_{2n}\}$ is a Cauchy sequence. In a similar way, we can prove that $\{x_{2n+1}\}$, $\{y_{2n}\}$, $\{y_{2n+1}\}$, $\{z_{2n}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences.

2.2.1 Tripled Fixed Point Theorems

By using the above lemmas, we shall prove the main theorem of Best Proximity Point:

Theorem 2.2.1.1 Let A and B be nonempty subsets of a metric space (X,d). The mappings $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ satisfying the following conditions:

- (i) (F,G) is a cyclic contraction pair of mappings;
- (ii) (A,B) and (B,A) satisfying the property UC^*.

If

$x_{2n+1} = F(x_{2n},y_{2n},z_{2n}), y_{2n+1} = F(y_{2n},z_{2n},x_{2n}), z_{2n+1} = F(z_{2n},x_{2n},y_{2n}),$

for all $n \in \mathbb{N} \cup \{0\}$,

and

$x_{2n+2} = G(x_{2n+1},y_{2n+1},z_{2n+1}), y_{2n+2} = G(y_{2n+1},z_{2n+1},x_{2n+1}), z_{2n+2} = G(z_{2n+1},x_{2n+1},y_{2n+1}),$

for all $n \in \mathbb{N} \cup \{0\}$,

then

(a) $\{x_{2n}\}, \{x_{2n+1}\}, \{y_{2n}\}, \{y_{2n+1}\}, \{z_{2n}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences.

(b) F and G have a tripled best proximity point, say respectively (p, q, r) and (p', q', r') such that

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B),$$

whenever, as $n \to \infty$, we have $x_{2n} \to p$, $y_{2n} \to q$, $z_{2n} \to r$, $x_{2n+1} \to p'$, $y_{2n+1} \to q'$ and $z_{2n+1} \to r'$.

Proof By Lemma 2.2.9, as $n \to \infty$, we get $d(x_{2n}, x_{2n+1}) \to d(A, B)$. Then, by Lemma 2.2.11, we have that $\{x_{2n}\}, \{y_{2n}\}$ and $\{z_{2n}\}$ are Cauchy sequences and hence, there exist $p, q, r \in A$ such that $x_{2n} \to p, y_{2n} \to q$ and $z_{2n} \to r$. From

$$d(A, B) \leq d(p, x_{2n-1}) \leq d(p, x_{2n}) + d(x_{2n}, x_{2n-1}), \tag{2.2.4}$$

letting $n \to \infty$ in (2.2.4), we have $d(p, x_{2n-1}) \to d(A, B)$. By a similar argument we get $d(q, y_{2n-1}) \to d(A, B)$ and $d(r, z_{2n-1}) \to d(A, B)$. It follows that

$$\begin{aligned} d(x_{2n}, F(p, q, r)) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(p, q, r)) \\ &\leq \tfrac{\alpha}{3}[d(x_{2n-1}, p) + d(y_{2n-1}, q) + d(z_{2n-1}, r)] + (1 - \alpha)d(A, B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p, q, r)) = d(A, B)$. Similarly, we can prove that $d(q, F(q, r, p)) = d(A, B)$ and $d(r, F(r, p, q)) = d(A, B)$. Consequently, (p, q, r) is a tripled best proximity point of F.

Similarly, we can prove that there exist $p', q', r' \in B$ such that $x_{2n+1} \to p'$, $y_{2n+1} \to q'$ and $z_{2n+1} \to r'$. Moreover, we also have $d(p', G(p', q', r')) = d(A, B)$, $d(q', G(q', r', p')) = d(A, B)$ and $d(r', G(r', p', q')) = d(A, B)$ and so (p', q', r') is a tripled best proximity point of G.

Finally, we show that $d(p, p') + d(q, q') + d(r, r') = 3d(A, B)$. For $n \in \mathbb{N} \cup \{0\}$, we have

$$d(x_{2n}, x_{2n+1}) = d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(x_{2n}, y_{2n}, z_{2n}))$$

$$\leq \tfrac{\alpha}{3}[d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n}) + d(z_{2n-1}, z_{2n})]$$

$$+(1-\alpha)d(A, B),$$

and letting $n \to \infty$, we get

$$d(p, p') \leq \frac{\alpha}{3}[d(p, p') + d(q, q') + d(r, r')] + (1-\alpha)d(A, B). \tag{2.2.5}$$

Analogously, for $n \in \mathbb{N} \cup \{0\}$, we have

$$d(y_{2n}, y_{2n+1}) = d(G(y_{2n-1}, z_{2n-1}, x_{2n-1}), F(y_{2n}, z_{2n}, x_{2n}))$$

$$\leq \tfrac{\alpha}{3}[d(y_{2n-1}, y_{2n}) + d(z_{2n-1}, z_{2n}) + d(x_{2n-1}, x_{2n})]$$

$$+(1-\alpha)d(A, B),$$

and letting $n \to \infty$, we get

$$d(q, q') \leq \frac{\alpha}{3}[d(q, q') + d(r, r') + d(p, p')] + (1-\alpha)d(A, B). \tag{2.2.6}$$

Again, for $n \in \mathbb{N} \cup \{0\}$, we have

$$d(z_{2n}, z_{2n+1}) = d(G(z_{2n-1}, x_{2n-1}, y_{2n-1}), F(z_{2n}, x_{2n}, y_{2n}))$$

$$\leq \tfrac{\alpha}{3}[d(z_{2n-1}, z_{2n}) + d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n})]$$

$$+(1-\alpha)d(A, B),$$

and letting $n \to \infty$, we get

$$d(r, r') \leq \frac{\alpha}{3}[d(r, r') + d(p, p') + d(q, q')] + (1 - \alpha)d(A, B). \tag{2.2.7}$$

It follows from (2.2.5), (2.2.6) and (2.2.7) that

$$d(p, p') + d(q, q') + d(r, r') \leq \frac{3\alpha}{3}[d(p, p') + d(q, q') + d(r, r')] + 3(1 - \alpha)d(A, B),$$

which implies that

$$d(p, p') + d(q, q') + d(r, r') \leq 3d(A, B). \tag{2.2.8}$$

Since we know that $d(A, B) \leq d(p, p')$, $d(A, B) \leq d(q, q')$ and $d(A, B) \leq d(r, r')$, so we have

$$3d(A, B) \leq d(p, p') + d(q, q') + d(r, r'). \tag{2.2.9}$$

From (2.2.8) and (2.2.9), we have

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B).$$

This completes the proof.

By using Lemma (2.2.9), we are ready to prove the following theorem.

Theorem 2.2.1.2 Let A and B be nonempty compact subsets of a metric space (X, d). The maps $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ be such that the ordered pair (F, G) is a cyclic contraction, if

$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}), y_{2n+1} = F(y_{2n}, z_{2n}, x_{2n}), z_{2n+1} = F(z_{2n}, x_{2n}, y_{2n}),$

for all $n \in \mathbb{N} \cup \{0\}$,

and

$x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}), y_{2n+2} = G(y_{2n+1}, z_{2n+1}, x_{2n+1}), z_{2n+2} = G(z_{2n+1}, x_{2n+1}, y_{2n+1}),$

for all $n \in \mathbb{N} \cup \{0\}$,

then the mappings F and G have a tripled best proximity point, say respectively (p, q, r) and (p', q', r'), such that

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B).$$

Proof. Using Lemma 2.2.9, we have $x_{2n}, y_{2n}, z_{2n} \in A$ and $x_{2n+1}, y_{2n+1}, z_{2n+1} \in B$ for all $n \in N \cup \{0\}$. Since A is compact, the sequences $\{x_{2n}\}, \{y_{2n}\}$ and $\{z_{2n}\}$ have convergent subsequences $\{x_{2n_k}\}, \{y_{2n_k}\}$ and $\{z_{2n_k}\}$ respectively, such that

$$x_{2n_k} \to p \in A, y_{2n_k} \to q \in A, \text{ and } z_{2n_k} \to r \in A,$$

as $n \to \infty$.

Clearly, we have

$$d(A, B) \leq d(p, x_{2n_k-1}) \leq d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \tag{2.2.10}$$

Again, by Lemma 2.2.9, we obtain $d(x_{2n_k}, x_{2n_k-1}) \to d(A, B)$ as $n \to \infty$. Now taking $k \to \infty$ in (2.2.10), we have $d(p, x_{2n_k-1}) \to d(A, B)$.

By a similar argument, we observe that $d(q, y_{2n_k-1}) \to d(A, B)$ and $d(r, z_{2n_k-1}) \to d(A, B)$.

Now, from

$$\begin{aligned} d(A, B) &\leq d((x_{2n_k}, F(p, q, r)) \\ &= d(G(x_{2n_k-1}, y_{2n_k-1}, z_{2n_k-1}), F(p, q, r)) \\ &\leq \tfrac{\alpha}{3}[d(x_{2n_k-1}, p) + d(y_{2n_k-1}, q) + d(z_{2n_k-1}, r)] + (1 - \alpha)d(A, B), \end{aligned}$$

letting $k \to \infty$, we get $d(p, F(p, q, r)) = d(A, B)$.

Similarly, we can prove that $d(q, F(q, r, p)) = d(A, B)$ and $d(r, F(r, p, q)) = d(A, B)$. Thus F has a tripled best proximity $(p, q, r) \in A \times A \times A$. Since B is also compact, so we can prove in a similar way that G has a tripled best proximity point $(p', q', r') \in$

$B \times B \times B$.

Finally, we show that $d(p, p') + d(q, q') + d(r, r') = 3d(A, B)$.

For $n \in \mathbb{N} \cup \{0\}$, we have

$$d(x_{2n_k}, x_{2n_k+1}) = d(G(x_{2n_k-1}, y_{2n_k-1}, z_{2n_k-1}), F(x_{2n_k}, y_{2n_k}, z_{2n_k}))$$

$$\leq \tfrac{\alpha}{3}[d(x_{2n_k-1}, x_{2n_k}) + d(y_{2n_k-1}, y_{2n_k}) + d(z_{2n_k-1}, z_{2n_k})]$$

$$+(1-\alpha)d(A, B),$$

and letting $n \to \infty$, we get

$$d(p, p') \leq \frac{\alpha}{3}[d(p, p') + d(q, q') + d(r, r')] + (1-\alpha)d(A, B). \tag{2.2.11}$$

Analogously, for $n \in \mathbb{N} \cup \{0\}$, we have

$$d(y_{2n_k}, y_{2n_k+1}) = d(G(y_{2n_k-1}, z_{2n_k-1}, x_{2n_k-1}), F(y_{2n_k}, z_{2n_k}, x_{2n_k}))$$

$$\leq \tfrac{\alpha}{3}[d(y_{2n_k-1}, y_{2n_k}) + d(z_{2n_k-1}, z_{2n_k}) + d(x_{2n_k-1}, x_{2n_k})]$$

$$+(1-\alpha)d(A, B),$$

and letting $n \to \infty$, we get

$$d(q, q') \leq \frac{\alpha}{3}[d(q, q') + d(r, r') + d(p, p')] + (1-\alpha)d(A, B). \tag{2.2.12}$$

Again, for $n \in \mathbb{N} \cup \{0\}$, we have

$$d(z_{2n_k}, z_{2n_k+1}) = d(G(z_{2n_k-1}, x_{2n_k-1}, y_{2n_k-1}), F(z_{2n_k}, x_{2n_k}, y_{2n_k}))$$

$$\leq \tfrac{\alpha}{3}[d(z_{2n_k-1}, z_{2n_k}) + d(x_{2n_k-1}, x_{2n_k}) + d(y_{2n_k-1}, y_{2n_k})]$$

$$+(1-\alpha)d(A, B),$$

and letting $n \to \infty$, we get

$$d(r, r') \leq \frac{\alpha}{3}[d(r, r') + d(p, p') + d(q, q')] + (1 - \alpha)d(A, B). \tag{2.2.13}$$

It follows from (2.2.11), (2.2.12) and (2.2.13) that

$$d(p, p') + d(q, q') + d(r, r') \leq \frac{3\alpha}{3}[d(p, p') + d(q, q') + d(r, r')] + 3(1 - \alpha)d(A, B),$$

which implies that

$$d(p, p') + d(q, q') + d(r, r') \leq 3d(A, B). \tag{2.2.14}$$

On the other hand, we know that $d(A, B) \leq d(p, p')$, $d(A, B) \leq d(q, q')$ and $d(A, B) \leq d(r, r')$, so we obtain by combining all to obtain the following condition:

$$3d(A, B) \leq d(p, p') + d(q, q') + d(r, r'). \tag{2.2.15}$$

Finally, from (2.2.14) and (2.2.15), we get

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B).$$

This completes the proof.

Now, here we shall consider the case of $d(A, B) = 0$ to prove the following result as a corollary to Theorem (2.2.1.2).

Corollary 2.2.1.3 Let A and B be nonempty compact subsets of a metric space (X, d), satisfying all the hypothesis of Theorem (2.2.1.2). Indeed, if $d(A, B) = 0$, then F and G have a unique common tripled fixed point $(p, q, r) \in A \cap B \times A \cap B \times A \cap B$.

Proof. If $d(A, B) = 0$ then (A, B) and (B, A) satisfying the property UC^*. Therefore, by using Theorem 2.2.2.1 we claim that F has a tripled best proximity point

$(p, q, r) \in A \times A \times A$ that is

$$d(p, F(p, q, r)) = d(q, F(q, r, p)) = d(r, F(r, p, q) = d(A, B) \qquad (2.2.16)$$

and G has a tripled best proximity point $(p', q', r') \in B \times B \times B$ that is

$$d(p', G(p', q', r')) = d(q', G(q', r', p')) = d(r', G(r', p', q')) = d(A, B). \qquad (2.2.17)$$

Also by Theorem 2.2.1.2, we have

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B). \qquad (2.2.18)$$

Assuming $d(A, B) = 0$ and using (2.2.16) we conclude that

$$p = F(p, q, r), q = F(q, r, p) \text{ and } r = F(r, p, q),$$

i.e. (p, q, r) is a tripled fixed point of F.

Again using (2.2.17) and applying $d(A, B) = 0$, we get

$$p' = G(p', q', r'), q' = G(q', r', p') \text{ and } r' = G(r', p', q'),$$

i.e. (p', q', r') is a tripled fixed point of G.

A similar argument for (2.2.18) and $d(A, B) = 0$, we have

$$d(p, p') + d(q, q') + d(r, r') = 0 \Rightarrow p = p', q = q' \text{ and } r = r'.$$

Consequently, we conclude that $(p, q, r) \in A \cap B \times A \cap B \times A \cap B$ is a common tripled fixed point of F and G.

To show the uniqueness of common tripled fixed point of F and G, suppose that $(\hat{p}, \hat{q}, \hat{r})$ is also a common tripled fixed point of F and G, i.e.

$$\hat{p} = F(\hat{p}, \hat{q}, \hat{r}) = G(\hat{p}, \hat{q}, \hat{r}), \hat{q} = F(\hat{q}, \hat{r}, \hat{p}) = G(\hat{q}, \hat{r}, \hat{p}) \text{ and } \hat{r} = F(\hat{r}, \hat{p}, \hat{q}) = G(\hat{r}, \hat{p}, \hat{q}).$$

Using the fact that the ordered pair (F, G) is a cyclic contraction, we get

$$d(p,\hat{p}) = d(F(p,q,r), G(\hat{p},\hat{q},\hat{r})) \leq \frac{\alpha}{3}[d(p,\hat{p}) + d(q,\hat{q}) + d(r,\hat{r})], \tag{2.2.19}$$

$$d(q,\hat{q}) = d(F(q,r,p), G(\hat{q},\hat{r},\hat{p})) \leq \frac{\alpha}{3}[d(q,\hat{q}) + d(r,\hat{r}) + d(p,\hat{p})], \tag{2.2.20}$$

$$d(r,\hat{r}) = d(F(r,p,q), G(\hat{r},\hat{p},\hat{q})) \leq \frac{\alpha}{3}[d(r,\hat{r}) + d(p,\hat{p}) + d(q,\hat{q})]. \tag{2.2.21}$$

Adding (2.2.19), (2.2.20) and (2.2.21), we obtain

$$d(p,\hat{p}) + d(q,\hat{q}) + d(r,\hat{r}) \leq \alpha[d(p,\hat{p}) + d(q,\hat{q}) + d(r,\hat{r})],$$

which implies that $d(p,\hat{p}) + d(q,\hat{q}) + d(r,\hat{r}) = 0 \Rightarrow d(p,\hat{p}) = d(q,\hat{q}) = d(r,\hat{r}) = 0$. Thus (p,q,r) is a unique common tripled fixed point of F and G in $A \cap B \times A \cap B \times A \cap B$.

Example 2.2.1.4 Let $X = \mathbb{R}$ endowed with the usual metric $d(x,y) =\mid x - y \mid$ and let $A = [-1, 0]$ and $B = [0, 1]$. Define $F : A \times A \times A \to B$ by

$$F(x,y,z) = \frac{-x-y-z}{6} \text{ and } G(u,v,w) = \frac{-u-v-w}{12}.$$

Clearly, $d(A,B) = 0$ and (F,G) is a cyclic contraction with $\alpha = \frac{1}{2}$. Indeed, for arbitrary $(x,y,z) \in A \times A \times A$ and $(u,v,w) \in B \times B \times B$, we get

$$\begin{aligned} d(F(x,y,z), G(u,v,w)) &=\mid \tfrac{-x-y-z}{6} - \tfrac{-u-v-w}{12} \mid \\ &\leq\mid \tfrac{-x-y-z}{6} + \tfrac{2u+2v+2w}{12} \mid \\ &= \tfrac{1}{6}(\mid x-u \mid + \mid y-v \mid + \mid z-w \mid) \\ &= \tfrac{\alpha}{3}[d(x,u) + d(y,v) + d(z,w)] + (1-\alpha)d(A,B). \end{aligned}$$

Therefore, all the hypotheses of Corollary 2.2.1.3 hold. Thus, F and G have a unique common tripled fixed point $(0,0,0) \in A \cap B \times A \cap B \times A \cap B$.

If we take $A = B$ and $F = G$ in Corollary 2.2.1.3, then we obtain the following consequence.

Corollary 2.2.1.5 Let A be a nonempty compact subset of a metric space (X, d) and $F : A \times A \times A \to A$ be such that

$$d(F(x, y, z), F(u, v, w)) \leq \frac{\alpha}{3}[d(x, u) + d(y, v) + d(z, w)], \qquad (2.2.22)$$

for all $(x, y, z), (u, v, w) \in A \times A \times A$ and some non-negative number $\alpha < 1$. Then F has a unique tripled fixed point $(p, q, r) \in A \times A \times A$.

2.2.2 Application in Integral Equations

Now, we shall establish an application in integral equation to our corollary(2.2.1.5):

$$x(t) = \int_0^T k(t, s)[f(s, x(s)) + g(s, x(s)) + h(s, x(s))]\ ds + b(t), \quad t \in [0, T]. \quad (2.2.23)$$

Let $C([0, T], \mathbb{R})$ be the set of continuous functions defined on $[0, T]$, where $T > 0$. Also define the metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, u) = \sup_{t \in [0,T]} |x(t) - u(t)|, \text{ for all } x, u \in \mathbb{R}.$$

Note that $(C([0, T], \mathbb{R}), d)$ is a complete metric space. Let Y be a compact subset of $(C([0, T], \mathbb{R}), d)$.

Now, we are ready to prove the following theorem:

Theorem 2.2.2.1 Suppose the following hypothesis are satisfied:

(h1) $k \in C([0, T] \times [0, T], \mathbb{R})$ and $\sup_{s,t \in [0,T]} |k(t, s)| = \delta < 1/T$;

(h2) $b \in Y$;

(h3) $f, g, h \in C([0,T] \times \mathbb{R}, \mathbb{R})$;

(h4) for all $x, y, z, u, v, w \in Y$ and $t \in [0,T]$ we have

$$\begin{aligned}&|f(t,x(t)) - f(t,u(t))| + |g(t,y(t)) - g(t,v(t))| + |h(t,z(t)) - h(t,w(t))| \\ &\leq \frac{1}{3}(|x(t) - u(t)| + |y(t) - v(t)| + |z(t) - w(t)|).\end{aligned}$$

Then, the integral equation (2.2.23) has a unique solution.

Proof. Define the mapping $F : Y \times Y \times Y \to Y$ by

$$F(x,y,z)(t) = \int_0^T k(t,s)[f(s,x(s)) + g(s,y(s)) + h(s,z(s))]\ ds + b(t), \quad t \in [0,T].$$

It is easy to show that (x, y, z) is a solution of (2.2.23) if and only if (x, y, z) is a tripled fixed point of F. Now, we intend to prove the existence and uniqueness of such a point, by using Corollary 2.2.1.5. Indeed, by hypothesis (h4) we get easily

$$\begin{aligned}&|F(x,y,z)(t) - F(u,v,w)(t)| \\ &\qquad \leq \textstyle\int_0^T |k(t,s)|\ \frac{1}{3}[|x(s) - u(s)| + |y(s) - v(s)| + |z(s) - w(s)|]\ ds \\ &\qquad \leq \textstyle\frac{1}{3}\left(\int_0^T |k(t,s)|\ ds\right)[d(x,u) + d(y,v) + d(z,w)],\end{aligned}$$

for all $x, y, z, u, v, w \in Y$ and $t \in [0,T]$.

By hypothesis (h1), it follows that

$$d(F(x,y,z), F(u,v,w)) \leq \frac{\delta T}{3}[d(x,u) + d(y,v) + d(z,w)],$$

for all $x, y, z, u, v, w \in Y$, that is condition (2.2.22) of Corollary 2.2.1.5 holds true with $\alpha = \delta T < 1$. Hence, by using Corollary 2.2.1.5, we obtain a unique tripled fixed point of F and so the integral equation (2.2.23) has a unique solution.

2.3 Section II

In this section, we improve the results of Babu and Alemayehu([60]) in metric spaces by giving shorter proof than of Babu and Alemayehu([60]).

We recall the following definitions to prove our theorems and propositions:

Definition 2.3.1([57]). Let A and S be selfmaps of a set X. If $A(u) = S(u) = w$ (say), $w \in X$, for some u in X, then u is called a ***coincidence point*** of A and S and the set of coincidence points of A and S in X is denoted by $C(A, S)$ and w is called a ***point of coincidence*** of A and S.

Definition 2.3.2 Let A, B, S and T be selfmaps of a set X. If $u \in C(A, S)$ and $v \in C(B, T)$ for some $u, v \in X$ and $A(u) = S(u) = B(v) = T(v) = z(say)$, then z is called a ***common point of coincidence of the pairs*** (A, S) and (B, T).

Definition 2.3.3 The pair (A, S) is said to be

(i) satisfy property $(E.A)$([109]) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = t$ for some t in X.

(ii) be compatible ([53]) if $\lim_{n\to\infty} d(AS(x_n), SA(x_n)) = 0$, for some t in X whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = t$.

(iii) be weakly compatible ([58]), if the commute at their coincidence point.

(iv) be occasionally weakly compatible (owc) ([111], [113]), if $AS(x) = SA(x)$ for some $x \in C(A, S)$.

Remark 2.3.4 (i) Every compatible pair is weakly compatible but not conversely([58]).

(ii) Weak compatibility and property (E. A) are independent to each other ([69]).

(iii) Every weakly compatible pair is occasionally weakly compatible but its converse need not be true ([114]).

(iv) Occasionally weakly compatible and property (E. A) are independent of each other ([59]).

Definition 2.3.5 ([169]) Let (X, d) be a metric space and A, B, S and T be four selfmaps on X. The pairs (A, S) and (B, T) are said to satisfy common property $(E.A)$, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} S(x_n) = t = \lim_{n\to\infty} B(y_n) = \lim_{n\to\infty} T(y_n)$ for some t in X.

Remark 2.3.6 Let A, B, S and T be self maps of a set X. If the pairs (A, S) and (B, T) have common point of coincidence in X then $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$. But converse is not true.

Example 2.3.7 Let $X = [0, 1]$ with usual metric and A, B, S and T self maps on X and defined by $A(x) = 1 - x^2; \quad S(x) = 1 - x; \quad B(x) = \frac{1}{2} + x^2; \quad T(x) = \frac{1+x}{2}$
for all $x \in X$.

When $x = 0$, then $A(0) = S(0) = 1$ and when $x = 1$, then $A(1) = S(1) = 0$.

When $x = 0$, then $B(0) = T(0) = \frac{1}{2}$ and when $x = \frac{1}{2}$, then $B(\frac{1}{2}) = T(\frac{1}{2}) = \frac{3}{4}$.

Then we observe that $C(A, S) = \{0, 1\}$ and $C(B, T) = \{\frac{1}{2}, \frac{3}{4}\}$ but the pairs (A, S) and (B, T) not having common point of coincidence.

Remark 2.3.8 The converse of the Remark 2.3.6 is true provided it satisfies inequality (2.3.2).

Proposition 2.3.9 ([110]) Let A and S be two self maps of a set X and the pair (A, S) is satisfies occasionally weakly compatible(owc) condition. If the pair (A, S) have unique point of coincidence $A(x) = S(x) = z$, then z is the unique common fixed point of A

and S.

Proof: Let

$$A(x) = S(x) = \{z\}(\text{say }) \qquad \text{for any } x \in C(A,S). \tag{2.3.1}$$

Assume the pair (A, S) satisfies the property owc, so that

$A(z) = AS(x) = SA(x) = S(z)$ implies that $z \in C(A, S)$.

From (2.3.1), we get $A(z) = S(z) = z$.

Hence proposition follows.

Tas, Telci, Fisher ([86]) proved the following:

Theorem 2.3.10 Let A, B, S and T be selfmaps of a complete metric space (X, d) such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and satisfy the inequality

$$[d(Ax, By)]^2 \leq c_1 \; max\{[d(Sx, Ax)]^2, [d(Ty, By)]^2, [d(Sx, Ty)]^2\}$$

$$+c_2 \; max\{d(Sx, Ax)d(Sx, By),\; d(Ty, Ax)d(Ty, By)\}$$

$$+c_3 d(Sx, By)d(Ty, Ax)$$

for all $x, y \in X$, where $c_1, c_2, c_3 \geq 0,\; c_1 + 2c_2 < 1,\; c_1 + c_3 < 1$. Further, assume that the pairs (A, S) and (B, T) are compatible on X. If one of the mappings A, B, S and T is continuous, then A, B, S and T have a unique common fixed point in X.

2.3.1 Main Results

First we establish the following propositions:

Proposition 2.3.1.1 Let A, B, S and T be self maps of a metric space (X, d) satisfying:

$$\begin{aligned}[d(Ax, By)]^2 \;\leq\;& c_1 \; max\{[d(Sx, Ax)]^2, [d(Ty, By)]^2, [d(Sx, Ty)]^2\} \\ & +c_2 \; max\{d(Sx, Ax)d(Sx, By),\; d(Ty, Ax)d(Ty, By)\} \\ & +c_3 d(Sx, By)d(Ty, Ax) \end{aligned} \tag{2.3.2}$$

for all $x, y \in X$, where $c_1, c_2, c_3 \geq 0$ and $c_1 + c_3 < 1$. Then the pairs (A, S) and (B, T) have a common point of coincidence in X if and only if $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$.

Proof. `If Part`: It is trivial.

`Only if part`: Assume $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$.

Then there is a $u \in C(A, S)$ and $v \in C(B, T)$ such that

$$A(u) = S(u) = p \text{ (say)} \tag{2.3.3}$$

$$B(v) = T(v) = q \text{ (say)} . \tag{2.3.4}$$

On taking $x = u$ and $y = v$ in (2.3.2), we get

$$[d(Au, Bv)]^2 \leq c_1 \, max\{[d(Su, Au)]^2, [d(Tv, Bv)]^2, [d(Su, Tv)]^2\}$$

$$+c_2 \, max\{d(Su, Au)d(Su, Bv), \; d(Tv, Au)d(Tv, Bv)\}$$

$$+c_3 d(Su, Bv)d(Tv, Au).$$

Using (2.3.3) and (2.3.4), we get

$[d(p, q)]^2 \leq (c_1 + c_3)[d(p, q)]^2$, a contradiction. Thus $p = q$.

Therefore A, B, S and T have common point of coincidence in X.

Proposition 2.3.1.2 Let A, B, S and T be four self maps of a metric space (X,d) satisfying the inequality (2.3.2). Suppose that either

(i) $B(X) \subseteq S(X)$, the pair (B, T) satisfies property (E.A.) and $T(X)$ is a closed subspace of X; or

(ii) $A(X) \subseteq T(X)$, the pair (A, S) satisfies property (E.A.) and $S(X)$ is a closed subspace of X, holds.

then,

(a) the pairs (A, S) and (B, T) satisfy the common property (E.A.), and

(b) the pairs (A, S) and (B, T) have a common point of coincidence in X.

Now, we are going to present the improved version of Theorem 2.2, ([60])

Theorem 2.3.1.3 Let A, B, S and T satisfy all the conditions given in Proposition 2.3.1.2 and the following additional assumption that the pairs (A, S) and (B, T) are owc on X.

Then A, B, S and T have a unique common fixed point in X.

Proof. By Proposition 2.3.1.2 the pairs (A, S) and (B, T) have common point of coincidence. Therefore there is $u \in C(A, S)$ and $v \in C(B, T)$ such that

$$A(u) = S(u) = z(say) = B(v) = T(v). \tag{2.3.5}$$

Now, we show that z is unique common point of coincidence of the pairs (A, S) and (B, T).

Let if possible z' is another point of coincidence of A, B, S and T. Then there is $u' \in C(A, S)$ and $v' \in C(B, T)$ such that

$$A(u') = S(u') = z'(say) = B(v') = T(v'). \tag{2.3.6}$$

Putting $x = u$ and $y = v'$ in inequality (2.3.2), we have

$[d(Au, Bv)]^2 \leq c_1 \; max\{[d(Su, Au)]^2, [d(Tv, Bv)]^2, [d(Su, Tv)]^2\}$

$+c_2\, max\{d(Su, Au)d(Su, Bv),\ d(Tv, Au)d(Tv, Bv)\}$

$+c_3 d(Su, Bv)d(Tv, Au)$

Now, using (2.3.5) and (2.3.6), we get

$[d(z, z')]^2 \leq (c_1 + c_3)[d(z, z')]^2$ and arrive at a contradiction. Hence $z = z'$ and we have $C(A, S) = \{z\} = C(B, T)$

By Proposition 2.3.9, z is the unique common fixed point of A, B, S and T in X.

Remark 2.3.1.4 Proposition 2.5 of ([60]) and Theorem 2.6 of ([60]) are remain true, if we replace completeness of $S(X)$ and $T(X)$ by the completeness of $S(X) \cap T(X)$ in X. For this we have given an Example 2.3.7 in the following manner without proof.

Now, we modify the Proposition 2.5 and Theorem 2.6 of ([60]).

Proposition 2.3.1.5 Let A, B, S and T be four self maps of a metric space (X, d) satisfying the inequality (2.3.2) of proposition 2.3.1.1 Suppose that (A, S) and (B, T) satisfy a common property $(E.A)$ and $S(X) \cap T(X)$ is a closed subspace of X, then A, B, S and T have unique common point of coincidence.

Theorem 2.3.1.6 In addition to the above proposition 2.3.1.5 on A, B, S and T, if both the pairs (A, S) and (B, T) are owc maps on X, then the point of coincidence is a unique common fixed point of A, B, S and T.

Proof. By Proposition 2.3.1.5 the pairs (A, S) and (B, T) have common point of coincidence. Therefore there is $u \in C(A, S)$ and $v \in C(B, T)$ such that

$$A(u) = S(u) = z(say) = B(v) = T(v). \tag{2.3.7}$$

Now, we show that z is unique common point of coincidence of the pairs (A, S) and

(B, T).

Let if possible z' is another point of coincidence of A, B, S and T. Then there is $u' \in C(A, S)$ and $v' \in C(B, T)$ such that

$$A(u') = S(u') = z'(say) = B(v') = T(v'). \tag{2.3.8}$$

Putting $x = u$ and $y = v'$ in inequality (2.3.2), we have

$$[d(Au, Bv)]^2 \leq c_1 \ max\{[d(Su, Au)]^2, [d(Tv, Bv)]^2, [d(Su, Tv)]^2\}$$

$$+c_2 \ max\{d(Su, Au)d(Su, Bv), \ d(Tv, Au)d(Tv, Bv)\}$$

$$+c_3 d(Su, Bv)d(Tv, Au)$$

Now, using (2.3.7) and (2.3.8), we get

$[d(z, z')]^2 \leq (c_1 + c_3)[d(z, z')]^2$ and arrive at a contradiction. Hence $z = z'$ and we have $C(A, S) = \{z\} = C(B, T)$

By Proposition 2.3.9, z is the unique common fixed point of A, B, S and T in X.

Now, we are ready to give an example to support our main Theorem 2.3.1.4.

Example 2.3.1.7 Let $X = [\frac{1}{3}, 1)$ with the usual metric. We define mappings A, B, S and T on X by

$$A(x) = \begin{cases} \frac{1}{3}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \frac{2}{3}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases} \qquad B(x) = \begin{cases} \frac{3}{4}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \frac{2}{3}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{2}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \\ \frac{1}{3} + \frac{x}{2}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases} \qquad T(x) = \begin{cases} \frac{5}{6}, & \text{if, } x \in [\frac{1}{3}, \frac{2}{3}); \\ \\ 1 - \frac{x}{2}, & \text{if, } x \in [\frac{2}{3}, 1) \end{cases}$$

We observe that $S(X) = \{\frac{1}{2}\} \cup [\frac{2}{3}, \frac{5}{6})$ and $T(X) = (\frac{1}{2}, \frac{2}{3}] \cup \{\frac{5}{6}\}$ are not closed and $S(X) \cap T(X) = \{\frac{2}{3}\}$ is a closed subspace of X.

The pairs (A, S) and (B, T) satisfies a common property (E.A) at the sequence $\{x_n\}$, $\{x_n\} = \frac{2}{3} + \frac{1}{n+3}, \quad n = 1, 2, 3, ...$ in X.

Case (i): If $x, y \in [\frac{1}{3}, \frac{2}{3})$ then the inequality (2.3.2), we get

$$(\tfrac{5}{12})^2 \le c_1 \ max\{(\tfrac{1}{6})^2, (\tfrac{1}{12})^2, (\tfrac{1}{3})^2\} + c_2 \ max\{\tfrac{1}{6}.\tfrac{1}{4}, \tfrac{1}{2}.\tfrac{1}{12}\} + c_3 \ \tfrac{1}{4}.\tfrac{1}{2}$$

$$i.e., \quad 25 \le 16c_1 + c_2 6 + c_3 18$$

Case (ii): If $x, y \in [\frac{2}{3}, 1)$ the inequality (2.3.2) holds trivial.

Case (iii): If $x \in [\frac{1}{3}, \frac{2}{3})$ and $y \in [\frac{2}{3}, 1)$ then from inequality (2.3.2), we have

$$(\tfrac{1}{3})^2 \le c_1 max\{\tfrac{1}{36}, (\tfrac{y}{2} - \tfrac{1}{3})^2, (\tfrac{1-y}{2})^2\} + c_2 max\{\tfrac{1}{36}, (\tfrac{2}{3} - \tfrac{y}{2})(\tfrac{y}{2} - \tfrac{1}{3})\} + c_3(\tfrac{2}{3} - \tfrac{y}{2}).$$

$$4 \le c_1 + c_2 + c_3.(4 - 3y).$$

Case (iv): if $x \in [\frac{2}{3}, 1)$ and $y \in [\frac{1}{3}, \frac{2}{3})$

$$(\tfrac{1}{12})^2 \le c_1 max\{(\tfrac{x}{2} - \tfrac{1}{3})^2, (\tfrac{1}{12})^2, (\tfrac{x-1}{2})^2\} + c_2 max\{| \tfrac{x}{2} - \tfrac{1}{3} || \tfrac{x}{2} - \tfrac{5}{12} |, \tfrac{1}{6}.\tfrac{1}{12}\}$$
$$+ c_3 \tfrac{1}{6} \mid \tfrac{x}{2} - \tfrac{5}{12} \mid$$

In all cases the inequality (2.3.2) holds with $c_1 = \frac{1}{3}, c_2 = 5\frac{5}{6}$ and $c_3 = \frac{1}{2}$. The pairs (A, S) and (B, T) satisfies owc at the point $\frac{2}{3}$. The point $\frac{2}{3}$ is a unique fixed point of A, B, S and T.

Chapter 3

TRIPLED COMMON FIXED POINT THEOREMS IN ORDERED CONE METRIC SPACES

3.1

In this chapter, we establish a tripled coincidence point theorem in ordered cone metric spaces over a solid cone. Our results extends coupled common fixed point theorems of Nashine, Kadelburg and Radenovic ([68]).

We recall some definitions for our main results.

Cone metric spaces:

Definition 3.1.1 Let E be a real Banach space with respect to a given norm $\| \, . \, \|_E$ and let 0_E be the zero vector of E. A non-empty subset P of E is called a cone, if the following conditions hold:

1. P is closed, non-empty and $P \neq \{0_E\}$;
2. $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow (ax + by) \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$.

A cone $P \subset E$ is partial ordering $\preceq$ with respect to P is naturally defined by $x \preceq y$ if and only if $y - x \in P$ for $x, y \in E$. We shall write $x \preceq y$ to indicate that $x \preceq y$ but

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$x \neq y$ while $x \ll y$ will stand for $y - x \in int(P), int(P)$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0_E \preceq x \preceq y$ implies $\parallel x \parallel_E \leq K \parallel y \parallel_E$.

We always suppose that E is a real Banach spaces with cone P satisfying $int(P) \neq \emptyset$ (such cones are called solid).

Definition 3.1.2 Let X be a nonempty set. Suppose the mapping $d : X \times X \to P$ satisfies:

1. $d(x, y) = 0_E$ if and only if $x = y$;

2. $d(x, y) = d(y, x)$ for all $x, y \in X$;

3. $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. It is obvious that cone metric spaces generalizes metric spaces.

Example 3.1.3 Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 3.1.4 Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(a) If for every $c \in E$ with $0_E \ll c$ there is $N \in \mathbf{N}$ such that $d(x_n, x) \ll c$, for all $n \geq N$, then $\{x_n\}$ is said to be convergent to x. This limit is denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) If for every $c \in E$ with $0_E \ll c$ there is $N \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$, for all $n, m \geq N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(c) If every Cauchy sequence in X is convergent in X, then (X, d) is called a complete cone metric spaces.

Let (X, d) be a cone metric space. Then the following properties are often used(particularly when dealing with cone metric spaces in which the cone need not be normal):

(P_1) If E is a real Banach space with a cone P and if $a \preceq ha$ where $a \in P$ and $h \in [0, 1)$, then $a = 0_E$;

(P_2) if $0_E \preceq u \ll c$ for each $0_E \ll c$, then $u = 0_E$;

(P_3) if $u, v, w \in E, u \preceq v$ and $v \ll w$, then $u \ll w$;

(P_4) if $c \in int(P), 0 \preceq a_n \in E$ and $a_n \to 0_E$, then there exists $k \in \mathbf{N}$ such that for all $n > k$ we have $a_n \ll c$.

In 2004, the concept of *partially ordered metric space*(POMS) was introduced by Ran and Reurings ([20]). Guo and Lakshmikantham ([37]) studied the concept of coupled fixed points in POMS. Later Bhaskar and Lakshmikantham ([154]) introduced the monotone property in POMS and their results were supported by an application to the existence of periodic boundary value problems. Recently, Karapinar([46]) proved coupled fixed point theorems for nonlinear contractions in ordered cone metric spaces over normal cones without regularity condition. He assumed continuity and commutativity for both maps in a complete POMS. Shatanawi ([167]) proved coupled coincidence and coupled fixed point theorems in cone metric spaces which were not necessarily normal. Some results on these spaces are given by Sabetghadam ([48]) , Ding and Li ([66]), and Aydi, Samet and Vetro([67]).
According to Borcut and Berinde ([157]), we recall the following definition(3.1.5 - 3.1.6):
Consider partial ordering on $X \times X \times X$ in the following manner:
for $(x, y, z), (u, v, w) \in X \times X \times X$, $(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w$.
Definition 3.1.5 Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$ and $g : X \to X$. The mapping F is said to have mixed g - monotone property if F is

monotone g - non-decreasing in x and z is monotone g - non-increasing in y that is, for any $x, y, z \in X$

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y, z) \leq F(x_2, y, z) \tag{3.1.1}$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1, z) \geq F(x, y_2, z) \tag{3.1.2}$$

$$z_1, z_2 \in X, g(z_1) \leq g(z_2) \Rightarrow F(x, y, z_1) \leq F(x, y, z_2) \tag{3.1.3}$$

hold.

Definition 3.1.6 Let $(X, d, \preceq)$ be a nonempty ordered cone metric space and $F : X \times X \times X \to X$, $g : X \to X$. An element $(x, y, z) \in X \times X \times X$ is called:

(T_1) a tripled coincidence point of mappings F and g if $F(x, y, z) = g(x)$, $F(y, x, y) = g(y)$, $F(z, y, x) = g(z)$.

(T_2) a tripled fixed point of the F if $F(x, y, z) = g(x) = x$, $F(y, x, y) = g(y) = y$ and $F(z, y, x) = g(z) = z$; in this case (gx, gy, gz) is called a triplet point of coincidence.

Definition 3.1.7 ([91]) Let $(X, d, \preceq)$ be a nonempty ordered cone metric space. Then we say that the mappings $F : X \times X \times X \to X$ and $g : X \to X$ are w - compatible if, $gF(x, y, z) = F(g(x), g(y), g(z))$, $gF(y, z, y) = F(g(y), g(z), g(y))$ and $gF(z, y, x) = F(g(z), g(y), g(x))$, holds, whenever $g(x) = F(x, y, z)$, $g(y) = F(y, z, y)$ and $g(z) = F(z, y, x)$.

3.1.1 Main Results

Theorem 3.1.1.1 Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone P. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that F has a mixed g -

monotone property on X and there exists three elements $x_0, y_0, z_0 \in X$ with

$$\begin{cases} g(x_0) \preceq F(x_0, y_0, z_0), \\ g(y_0) \succeq F(y_0, z_0, y_0) \\ g(z_0) \preceq F(z_0, x_0, y_0). \end{cases} \tag{3.1.4}$$

Suppose further that F and g satisfy

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) \quad \preceq \quad & a_1 d(gx, gu) + a_2 d(F(x,y,z), gx) + a_3 d(gy, gv) \\ & + a_4 d(F(u,v,w), gu) + a_5 d(F(x,y,z), gu) \\ & + a_6 d(F(u,v,w), gx), \end{aligned} \tag{3.1.5}$$

for all $(x,y,z), (u,v,w) \in X \times X \times X$ with $(gx \preceq gu,\ gy \succeq gv \quad \text{and} \quad gz \preceq gw)$, where $a_i \geq 0$, for $i = 1, 2, ..., 6$ and $\sum_{i=1}^{6} a_i < 1$. Further suppose that

$$F(X \times X \times X) \subseteq g(X); \tag{3.1.6}$$

$$g(X) \text{ is a complete subspaces of } X. \tag{3.1.7}$$

Also, suppose that if X has the following properties:

(i) $\{x_n\}$ is a non - decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in N$,

(ii) $\{y_n\}$ is a non - increasing sequence in X such that $y_n \to y$, then $y_n \succeq y$ for all $n \in N$,

(iii) $\{z_n\}$ is a non - decreasing sequence in X such that $z_n \to z$, then $z_n \preceq z$ for all $n \in N$, then there exist $x, y, z \in X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, z, y)$ and $g(z) = F(z, x, y)$, that is, F and g have a tripled coincidence point in X.

Proof.

Let $x_0, y_0, z_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0, z_0)$, $g(y_0) \succeq F(y_0, z_0, y_0)$ and $g(z_0) \preceq F(z_0, x_0, y_0)$.

Since $F(X \times X \times X) \subseteq g(X)$, we can define $x_1, y_1, z_1 \in X$ such that

$$g(x_1) = F(x_0, y_0, z_0), \quad g(y_1) = F(y_0, z_0, y_0) \quad \text{and} \quad g(z_1) = F(z_0, x_0, y_0).$$

For $x_2, y_2, z_2 \in X$ we can define

$$g(x_2) = F(x_1, y_1, z_1), g(y_2) = F(y_1, z_1, y_1) \quad \text{and} \quad g(z_2) = F(z_1, x_1, y_1).$$

Proceeding in this way, we can construct three sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n, z_n), g(y_{n+1}) = F(y_n, z_n, y_n), g(z_{n+1}) = F(z_n, x_n, y_n). \tag{3.1.8}$$

for all $n \geq 0$.

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}), g(y_n) \succeq g(y_{n+1}) \text{ and } g(z_n) \preceq g(z_{n+1}). \tag{3.1.9}$$

By condition (3.1.5), we have $g(x_0) \preceq F(x_0, y_0, z_0) = g(x_1)$, $g(y_0) \succeq F(y_0, z_0, y_0) = g(y_1)$ and $g(z_0) \preceq F(z_0, x_0, y_0) = g(z_1)$. i.e., (3.1.9) holds for $n = 0$. We assume that (3.1.9) holds for some $n > 0$. As F has the mixed g - monotone property and $g(x_n) \preceq g(x_{n+1}), g(y_n) \succeq g(y_{n+1})$ and $g(z_n) \preceq g(z_{n+1})$, from (3.1.8) and (3.1.1 -

3.1.3), we get

$$\begin{cases} g(x_{n+1}) = F(x_n, y_n, z_n) \preceq F(x_{n+1}, y_n, z_n), \\ g(y_{n+1}) = F(y_n, z_n, y_n) \succeq F(y_{n+1}, z_n, y_n), \\ g(z_{n+1}) = F(z_n, x_n, y_n) \preceq F(z_{n+1}, x_n, y_n). \end{cases} \tag{3.1.10}$$

Also, we have

$$\begin{cases} g(x_{n+2}) = F(x_{n+1}, y_{n+1}, z_{n+1}) \preceq F(x_{n+1}, y_n, z_n), \\ g(y_{n+2}) = F(y_{n+1}, z_{n+1}, y_{n+1}) \succeq F(y_{n+1}, z_n, y_n), \\ g(z_{n+2}) = F(z_{n+1}, x_{n+1}, y_{n+1}) \preceq F(z_{n+1}, x_n, y_n). \end{cases} \tag{3.1.11}$$

Then from (3.1.10) and (3.1.11) we obtain

$g(x_{n+1}) \preceq g(x_{n+2}), g(y_{n+1}) \succeq g(y_{n+2})$ and $g(z_{n+2}) \preceq g(z_{n+2})$.

By induction, we conclude that (3.1.9) holds for all $n \geq 0$.

Using condition (3.1.5), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_{n-1})) + a_3 d(gy_{n-1}, gy_n) \\ &\quad + a_4 d(F(x_n, y_n, z_n), gx_n)) + a_5 d(F(x_{n-1}, y_{n-1}, z_{n-1}), gx_n)) \\ &\quad + a_6 d(F(x_n, y_n, z_n), gx_{n-1})) \\ &= a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(gx_{n+1}, gx_n) \\ &\quad + a_5 d(gx_n, gx_n) + a_6 d(gx_{n+1}, gx_{n-1}) \\ &\preceq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(gx_{n+1}, gx_n) \\ &\quad + a_6 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]. \\ &= (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + (a_4 + a_6) d(gx_n, gx_{n+1}) + a_3 d(gy_{n-1}, gy_n) \end{aligned}$$

which implies that

$$(1-a_4-a_6)d(gx_n, gx_{n+1}) \preceq (a_1+a_2+a_6)d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) \quad (3.1.12)$$

Similarly,

$$d(gy_n, gy_{n+1}) = d(F(y_{n-1}, z_{n-1}, y_{n-1}, F(y_n, z_n, y_n))$$

$$\preceq a_1 d(gy_{n-1}, gy_n) + a_2 d(F(y_{n-1}, z_{n-1}, y_{n-1}, gy_{n-1})) + a_3 d(gz_{n-1}, gz_n)$$

$$+a_4 d(F(y_n, z_n, y_n, gy_n)) + a_5 d(F(y_{n-1}, z_{n-1}, y_{n-1}, gy_n))$$

$$+a_6 d(F(y_n, z_n, x_n, gy_{n-1}))$$

$$= a_1 d(gy_{n-1}, gy_n) + a_2 d(gy_n, gy_{n-1}) + a_3 d(gz_{n-1}, gz_n) + a_4 d(gy_{n+1}, gy_n)$$

$$+a_5 d(gy_n, gy_n) + a_6 d(gy_{n+1}, gy_{n-1})$$

$$\preceq a_1 d(gy_{n-1}, gy_n) + a_2 d(gy_n, gy_{n-1}) + a_3 d(gz_{n-1}, gz_n) + a_4 d(gy_{n+1}, gy_n)$$

$$+a_6[d(gy_{n+1}, gy_n) + d(gy_n, gy_{n-1})].$$

$$= (a_1+a_2+a_6)d(gy_{n-1}, gy_n) + (a_4+a_6)d(gy_n, gy_{n+1}) + a_3 d(gz_{n-1}, gz_n)$$

which implies that

$$(1-a_4-a_6)d(gy_n, gy_{n+1}) \preceq (a_1+a_2+a_6)d(gy_{n-1}, gy_n) + a_3 d(gz_{n-1}, gz_n) \quad (3.1.13)$$

Also,

$$d(gz_n, gz_{n+1}) = d(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n))$$

$$\preceq a_1 d(gz_{n-1}, gz_n) + a_2 d(F(z_{n-1}, x_{n-1}, y_{n-1}, gz_{n-1})) + a_3 d(gx_{n-1}, gx_n)$$

$$+a_4 d(F(z_n, x_n, y_n, gz_n)) + a_5 d(F(z_{n-1}, x_{n-1}, y_{n-1}, gz_n))$$

$$+a_6 d(F(z_n, x_n, y_n, gz_{n-1}))$$

$$= a_1 d(gz_{n-1}, gz_n) + a_2 d(gz_n, gz_{n-1}) + a_3 d(gx_{n-1}, gx_n) + a_4 d(gz_{n+1}, gz_n)$$

$$+a_5 d(gz_n, gz_n) + a_6 d(gz_{n+1}, gz_{n-1})$$

$$\preceq a_1 d(gz_{n-1}, gz_n) + a_2 d(gz_n, gz_{n-1}) + a_3 d(gx_{n-1}, gx_n) + a_4 d(gz_{n+1}, gz_n)$$

$$+a_6[d(gz_{n+1}, gz_n) + d(gz_n, gz_{n-1})]$$

$$= (a_1 + a_2) d(gz_{n-1}, gz_n) + a_3 d(gx_{n-1}, gx_n) + a_4 d(gz_{n+1}, gz_n)$$

$$+a_6[d(gz_{n+1}, gz_n) + d(gz_n, gz_{n-1})]$$

$$= (a_1 + a_2 + a_6) d(gz_{n-1}, gz_n) + (a_4 + a_6) d(gz_{n+1}, gz_n) + a_3 d(gx_{n-1}, gx_n)$$

which implies that

$$(1 - a_4 - a_6) d(gz_n, gz_{n+1}) \preceq (a_1 + a_2 + a_6) d(gz_{n-1}, gz_n) + a_3 d(gx_{n-1}, gx_n) \quad (3.1.14)$$

Adding (3.1.12), (3.1.13) and (3.1.14) we obtain that

$$(1 - a_4 - a_6)[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})]$$
$$\preceq (a_1 + a_2 + a_6)[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)]$$
$$+a_3[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)]$$
$$= (a_1 + a_2 + a_3 + a_6)[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)] \quad (3.1.15)$$

Now starting from $d(gx_{n+1}, gx_n) = d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}))$ and using that $g(x_{n-1}) \preceq g(x_n), g(y_{n-1}) \succeq g(y_n)$ and $g(z_{n-1}) \preceq g(z_n)$,

we get that

$$d(gx_{n+1}, gx_n) = d(F(x_n, y_n, z_n, F(x_{n-1}, y_{n-1}, z_{n-1}))$$

$$\preceq a_1 d(gx_n, gx_{n-1}) + a_2 d(F(x_n, y_n, z_n, gx_n)) + a_3 d(gy_n, gy_{n-1})$$

$$+a_4 d(F(x_{n-1}, y_{n-1}, z_{n-1}, gx_n)) + a_5 d(F(x_n, y_n, z_n, gx_{n-1}))$$

$$+a_6 d(F(x_{n-1}, y_{n-1}, z_{n-1}, gx_n))$$

$$= a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(gx_n, gx_{n-1})$$

$$+a_5 d(gx_{n+1}, gx_{n-1}) + a_6 d(gx_n, gx_n))$$

$$= a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(gx_n, gx_{n-1})$$

$$+a_5 [d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]$$

$$d(gx_{n+1}, gx_n) = (a_1 + a_4 + a_5) d(gx_n, gx_{n-1}) + (a_2 + a_5) d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) \tag{3.1.16}$$

Similarly,

$$\begin{aligned} d(gy_{n+1}, gy_n) &= d(F(y_n, z_n, y_n, F(y_{n-1}, z_{n-1}, y_{n-1})) \\ &\preceq (a_1 + a_4 + a_5) d(gy_n, gy_{n-1}) + (a_2 + a_5) d(gy_{n+1}, gy_n) \\ &\quad +a_3 d(gz_n, gz_{n-1}) \end{aligned} \tag{3.1.17}$$

$$\begin{aligned} d(gz_{n+1}, gz_n) &\preceq (a_1 + a_4 + a_5) d(gz_n, gz_{n-1}) + (a_2 + a_5) d(gz_{n+1}, gz_n) \\ &\quad +a_3 d(gx_n, gx_{n-1}) \end{aligned} \tag{3.1.18}$$

Combining (3.1.16), (3.1.17) and (3.1.18) we obtain that

$$(1 - a_2 - a_5)[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)]$$

$$\preceq (a_1 + a_4 + a_5)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})]$$

$$+a_3[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})]$$

$$= (a_1 + a_3 + a_4 + a_5)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})] \quad (3.1.19)$$

Again combining (3.1.15) and (3.1.19), we get

$$[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)]$$

$$\preceq \left(\frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6}{2 - a_2 - a_4 - a_5 - a_6}\right)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})]$$

$$= \lambda[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})]$$

where $\lambda = \dfrac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6}{2 - a_2 - a_4 - a_5 - a_6}$ and $0 \leq \lambda < 1$.
Since $\sum_{i=1}^{6} a_i < 1$. using the above relation n - times, we obtain

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)$$

$$\preceq \lambda[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) + d(gz_{n-1}, gz_n)]$$

$$\preceq \lambda^2[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1}) + d(gz_{n-2}, gz_{n-1})]$$

$$\vdots$$

$$\preceq \lambda^n[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)].$$

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n) \preceq \lambda^n[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)]. \quad (3.1.20)$$

As letting $n \to \infty$ in (3.1.20), we obtain

$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n) \to 0_E$ as $n \to \infty$.

Hence, $d(gx_{n+1}, gx_n) = d(gy_{n+1}, gy_n) = d(gz_{n+1}, gz_n) \to 0_E$ as $n \to \infty$.
Now we will show that $\{gx_n\}, \{gy_n\}$and$\{gz_n\}$ are Cauchy sequences. For any $m > n \geq 1$, and repeated use of triangle inequality, we have

$$
\begin{aligned}
&d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m) \\
&\qquad \preceq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\
&\qquad\quad + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) + \cdots + d(gy_{m-1}, gy_m) \\
&\qquad\quad + d(gz_n, gz_{n+1}) + d(gz_{n+1}, gz_{n+2}) + \cdots + d(gz_{m-1}, gz_m) \\
&\qquad \preceq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}][d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \\
&\qquad \preceq \tfrac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \\
&\qquad \to 0_E \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

from (P_4) it follows that for $0_E \ll c$, and for very large $n : \frac{\lambda^n}{1-\lambda}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gz_0, gz_1)] \ll c$, and by (P_3), it implies
$[d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)] \ll c$.
Since,

$$d(gx_n, gx_m) \preceq [d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)],$$

$$d(gy_n, gy_m) \preceq [d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)],$$

and

$$d(gz_n, gz_m) \preceq [d(gx_n, gx_m) + d(gy_n, gy_m) + d(gz_n, gz_m)]$$

again by $(P_3), d(gx_n, gx_m) \ll c, d(gy_n, gy_m) \ll c$ and $d(gz_n, gz_m) \ll c$ for very large enough n. Hence $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $g(X)$. By completeness of $g(X)$, there exists $gx, gy, gz \in g(X)$ such that $\{gx_n\} \to gx$, $\{gy_n\} \to gy$

and $\{gz_n\} \to gz$ as $n \to \infty$.

Since $\{gx_n\}$ and $\{gz_n\}$ are nondecreasing and $\{gy_n\}$ is non-increasing, using the conditions (i), (ii) and (iii), we have $g(x_n) \preceq gx$, $g(y_n) \succeq gy$ and $g(z_n) \preceq gz$ for all $n \geq 0$. If $g(x_n) = gx$, $g(y_n) = gy$ and $g(z_n) = gz$ for some $n \geq 0$, then $gx = g(x_n) \preceq g(x_{n+1}) \preceq gx = g(x_n)$, $gy \preceq g(y_{n+1}) \preceq g(y_n) = gy$ and $gz = g(z_n) \preceq g(z_{n+1}) \preceq gz = g(z_n)$ which implies that $gx = g(x_n) = F(x_n, y_n, z_n)$, $gy = g(y_n) = F(y_n, z_n, y_n)$, and $gz = g(z_n) = F(z_n, x_n, y_n)$, that is, (x_n, y_n, z_n) is a triplet coincidence point of F and g. Now, we suppose that $(gx_n, gy_n, gz_n) \neq (gx, gy, gz)$ for all $n \geq 0$.

Now we shall show that $F(x, y, z) = gx$, $F(y, z, y) = gy$ and $F(z, x, y) = gz$.

For this, consider

$$\begin{aligned}
d(F(x,y,z), gx) &\preceq d(F(x,y,z), gx_{n+1}) + d(gx_{n+1}, gx) \\
&= d(F(x,y,z), F(x_n, y_n, z_n)) + d(gx_{n+1}, gx) \\
&\preceq a_1 d(gx, gx_n) + a_2 d(F(x,y,z), gx) + a_3 d(gy, gy_n) + a_4 d(F(x_n, y_n, z_n), gx_n) \\
&\quad + a_5 d(F(x,y,z), gx_n) + a_6 d(F(x_n, y_n, z_n), gx) + d(gx_{n+1}, gx) \\
&= a_1 d(gx, gx_n) + a_2 d(F(x,y,z), gx) + a_3 d(gy, gy_n) + a_4 d(F(gx_{n+1}, gx_n) \\
&\quad + a_5 d(F(x,y,z), gx_n) + a_6 d(gx_{n+1}, gx_n) + a_6 d(gx_n, gx) \\
&\quad + d(gx_{n+1}, gx_n) + d(gx_n, gx)
\end{aligned}$$

$$\begin{aligned}
(1 - a_2 - a_5) d(F(x,y,z), gx) &\preceq (1 + a_1 + a_5 + a_6) d(gx_n, gx) + (1 + a_4 + a_6) d(gx_{n+1}, gx_n) \\
&\quad + a_3 d(gy, gy_n)
\end{aligned}$$

which further implies that

$$d(F(x,y,z), gx) \preceq \frac{1 + a_1 + a_5 + a_6}{(1 - a_2 - a_5)} d(gx, gx_n) + \frac{a_3}{(1 - a_2 - a_5)} d(gy, gy_n)$$

$$+\frac{1+a_4+a_6}{(1-a_2-a_5)}d(gx_{n+1},gx_n);$$

since $g(x_n)\to gx$, $g(y_n)\to gy$ and $g(z_n)\to gz$, then for $0_E\ll c$ there exists $N\in\mathbf{N}$ such that $d(gx,gx_n)\ll\frac{(1+a_1+a_5+a_6)}{(1-a_2-a_5)}c$, $d(gy,gy_n)\ll\frac{(a_3)}{(1-a_2-a_5)}c$, and $d(gx_{n+1},gx_n)\ll\frac{(1+a_4+a_6)}{(1-a_2-a_5)}c$ and for all $n\geq N$.

Thus, $d(F(x,y,z),gx)\ll c$.

Now according to (P_2) it follows that $d(F(x,y,z),gx)=0_E$, and $F(x,y,z)=gx$.

Similarly, we can get $F(y,z,y)=gy$ and $F(z,x,y)=gz$.

Hence (gx,gy,gz) is tripled coincidence point of mappings F and g.

This completes the proof.

Now we prove the existence and uniqueness of a tripled common fixed point.

Theorem 3.1.1.2 Let $(X,d,\preceq)$ be an ordered cone metric space over a solid cone P. Let $F:X\times X\times X\to X$ and $g:X\to X$ be two mappings satisfying the hypothesis of Theorem 3.1.1.2. Further assume that for every $(x,y,z),(x^*,y^*,z^*)\in X\times X\times X$ there exists $(u,v,w)\in X\times X\times X$ such that $(F(u,v,w),F(v,w,u),F(w,u,v))$ is comparable to $(F(x,y,z),F(y,z,x),F(z,x,y))$ and $(F(x^*,y^*,z^*),F(y^*,z^*,x^*),(z^*,x^*,y^*))$. Then F and g have a unique triple common fixed point, that is, there exists a unique $(u,v,w)\in X\times X\times X$ such that $u=g(u)=F(u,v,w)$, $g(v)=F(v,w,v)$ and $g(w)=F(w,u,v)$, provided F and g are w - compatible.

Proof. From Theorem 3.1.1.1, the set of tripled coincidence points of F and g is nonempty. Suppose (x,y,z) and (x^*,y^*,z^*) are tripled coincidence points of F, that is $g(x)=F(x,y,z)$, $g(y)=F(y,z,y)$, $g(z)=F(z,x,y)$, $g(x^*)=F(x^*,y^*,z^*)$, $g(y^*)=F(y^*,z^*,y^*)$ and $g(z^*)=F(z^*,x^*,y^*)$. We will prove that

$$g(x)=g(x^*),g(y)=g(y^*)\quad\text{and}\quad g(z)=g(z^*).$$

By assumption, there exists $(u,v,w)\in X\times X\times X$ such that $(F(u,v,w),F(v,w,u),F(w,u,v))$ is comparable to $(F(x,y,z),F(y,z,x),F(z,x,y))$ and $(F(x^*,y^*,z^*),F(y^*,z^*,x^*),(z^*,x^*,y^*))$.

Put $u_0 = u, v_0 = v, w_0 = w$ and choose $u_1, v_1, w_1 \in X$ so that $g(u_1) = F(u_0, v_0, w_0), g(v_1) = F(v_0, w_0, v_0)$ and $g(w_1) = F(w_0, u_0, v_0)$. Then, similarly as in the proof of Theorem 3.1.1.1, we can inductively define sequences $\{gu_n\}, \{gv_n\}$ and $\{gw_n\}$ with

$$g(u_{n+1}) = F(u_n, v_n, w_n), g(v_{n+1}) = F(v_n, w_n, v_n) \quad \text{and} \quad g(w_{n+1}) = F(w_n, u_n, v_n) \quad \forall \quad n.$$

Further, set $x_0 = x, y_0 = y, z_0 = z, x_0^* = x^*, y_0^* = y^*$ and $z_0^* = z^*$ and in a similar way, define the sequence $\{gx_n\}, \{gy_n\}, \{gz_n\}$ and $\{gx_n^*\}, \{y_n^*\}, \{z_n^*\}$. Then it is easy to show that

$$g(x_n) \to F(x, y, z), g(y_n) \to F(y, z, y) \quad \text{and} \quad g(z_n) \to F(z, x, y)$$

and

$$g(x_n^*) \to F(x^*, y^*, z^*), g(y_n^*) \to F(y^*, z^*, y^*) \quad \text{and} \quad g(z_n^*) \to F(z^*, x^*, y^*)$$

as $n \to \infty$. Since $(gx, gy, gz) = (F(x, y, z), F(y, z, y), F(z, x, y)) = (gx_1, gy_1, gz_1)$ and $(F(u, v, w), F(v, w, v), F(w, u, v)) = (gu_1, gv_1, gw_1)$ are comparable, then $gx \preceq gu_1, gy \succeq gv_1$ and $gz \preceq gw_1$. It is easy to show that, similarly, (gx, gy, gz) and (gu_n, gv_n, gw_n) are comparable for all $n \geq 1$, that is, $gx \preceq gu_n, gy \succeq gv_n$ and $gz \preceq gw_n$, or vice versa. Thus from (3.1.5) we have

$$\begin{aligned}
d(gx, gu_{n+1}) &= d(F(x, y, z), F(u_n, v_n, w_n)) \\
&\preceq a_1 d(gx, gu_n) + a_2 d(F(x, y, z, gx)) + a_3 d(gy, gv_n) \\
&\quad + a_4 d(F(u_n, v_n, w_n, gu_n)) + a_5 d(F(x, y, z, gu_n)) \\
&\quad + a_6 d(F(u_n, v_n, w_n, gx)) \\
&= a_1 d(gx, gu_n) + a_2 d(gx, gx) + a_3 d(gy, gv_n) + a_4 d(gu_{n+1}, gu_n) \\
&\quad + a_5 d(gx, gu_n) + a_6 d(gu_{n+1}, gx). \\
&= a_1 d(gx_1, gu_n) + a_3 d(gy, gv_n) + a_4 [d(gu_{n+1}, gx)
\end{aligned}$$

$$+d(gx, gx_n)] + a_6 d(gu_{n+1}, gx).$$

$$= (a_1 + a_4 + a_5) d(gx, gu_n) + a_3 d(gy, gv_n) + (a_4 + a_6) d(gu_{n+1}, gx).$$

which implies that

$$(1 - a_4 - a_6) d(gx, gu_{n+1}) \preceq (a_1 + a_4 + a_5) d(gx, gu_n) + a_3 d(gy, gv_n). \tag{3.1.21}$$

Similarly,

$$(1 - a_4 - a_6) d(gy, gv_{n+1}) \preceq (a_1 + a_4 + a_5) d(gy, gv_n) + a_3 d(gz, gw_n). \tag{3.1.22}$$

$$(1 - a_4 - a_6) d(gz, gw_{n+1}) \preceq (a_1 + a_4 + a_5) d(gz, gw_n) + a_3 d(gx, gu_n). \tag{3.1.23}$$

Adding up (3.1.21, 3.1.22) and (3.1.23) we obtain that

$$\begin{aligned}
&(1 - a_4 - a_6)[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \\
&\preceq (a_1 + a_4 + a_5)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
&+ a_3[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
&= (a_1 + a_3 + a_4 + a_5)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \qquad (3.1.24)
\end{aligned}$$

Now stating from $d(gu_{n+1}, gx) = d(F(u_n, v_n, w_n), F(x, y, z))$ we get that

$$d(gu_{n+1}, gx) = d(F(u_n, v_n, w_n, F(x, y, z))$$

$$\preceq a_1 d(gu_n, gx) + a_2 d(F(u_n, v_n, w_n, gu_n)) + a_3 d(gv_n, gy)$$

$$+a_4 d(F(x, y, z, gx)) + a_5 d(F(u_n, v_n, w_n, gx))$$

$$+a_6 d(F(x, y, z, gu_n))$$

$$= a_1 d(gu_n, gx) + a_2 d(gu_{n+1}, gu_n) + a_3 d(gv_n, gy) + a_4 d(gx, gx)$$

$+a_5d(gu_{n+1}, gx) + a_6d(gx, gu_n))$

$= a_1d(gu_n, gx) + a_2[d(gu_{n+1}, gx) + d(gx, gu_n)] + a_3d(gv_n, gy)$

$+a_4d(gx, gx) + a_5d(gu_{n+1}, gx) + a_6d(gx, gu_n))$

$$(1 - a_2 - a_5)d(gu_{n+1}, gx) = (a_1 + a_2 + a_6)d(gx, gu_n) + a_3d(gv_n, gy) \tag{3.1.25}$$

Similarly,

$$(1 - a_2 - a_5)d(gv_{n+1}, gy) = (a_1 + a_2 + a_6)d(gy, gv_n) + a_3d(gw_n, gz) \tag{3.1.26}$$

$$(1 - a_2 - a_5)d(gw_{n+1}, gz) = (a_1 + a_2 + a_6)d(gz, gw_n) + a_3d(gu_n, gx) \tag{3.1.27}$$

Adding (3.1.25), (3.1.26) and (3.1.27) we obtain that

$$\begin{aligned}
&(1 - a_2 - a_5)[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \\
&\preceq (a_1 + a_2 + a_6)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
&+a_3[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \\
&= (a_1 + a_2 + a_3 + a_6)[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \qquad (3.1.28)
\end{aligned}$$

Now adding (3.1.24) and (3.1.28) we get

$$\begin{aligned}
&[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \\
&\preceq (\frac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6}{2 - a_2 - a_4 - a_5 - a_6}) \\
&[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] \qquad (3.1.29)
\end{aligned}$$

with $0 \leq \lambda < 1$. Where $\lambda = \dfrac{2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6}{2 - a_2 - a_4 - a_5 - a_6}$.

Since $\sum_{i=1}^{6} a_i < 1$. using relation (3.1.29) n - times, we obtain

$$[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \preceq \lambda[d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)]$$

$$\preceq \lambda^2[d(gx, gu_{n-1}) + d(gy, gv_{n-1}) + d(gz, gw_{n-1})]$$

$$\vdots$$

$$\preceq \lambda^n[d(gx, gu_0) + d(gy, gv_0) + d(gz, gw_0)].$$

Then $[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \to 0_E$ as $n \to \infty$.

Thus $d(gx, gu_{n+1}) = d(gy, gv_{n+1}) = d(gz, gw_{n+1}) \to 0_E$ as $n \to \infty$. Since $0 \leq \lambda < 1$.

Hence $[d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})] \ll c$ for each $c \in intP$ and large n. Since $0_E \preceq d(gx, gu_{n+1}) \preceq [d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})]$, it follows by (P_3) that $d(gu_{n+1}, gx) \ll c$, for n large enough and so $\{gu_n\} \to gx$ when $n \to \infty$. Similarly, $\{gv_{n+1}\} \to gy$ and $\{gw_{n+1}\} \to gz$. By the same procedure one can show that $\{gu^*_{n+1}\} \to gx^*, \{gv^*_{n+1}\} \to gy^*$ and $\{gw^*_{n+1}\} \to gz^*$. By the uniqueness of the limit, it follows that $gx = gx^*, gy = gy^*$ and $gz = gz^*$. as $n \to \infty$. Hence (gx, gy, gz) is the unique tripled point of coincidence of F and g.

Now we show that F and g have a unique common tripled fixed point. For this, let $gx = u$. Then we have $u = gx = F(x, y, z)$. By w - compatibility of F and g, we have

$$gu = g(gx) = g(F(x, y, z)) = F(gx, gy, gz) = F(u, v, w),$$

$$gv = g(gy) = g(F(y, z, y)) = F(gy, gz, gy) = F(v, w, v), \text{ and}$$

$$gw = g(gz) = g(F(z, x, y)) = F(gz, gy, gx) = F(w, u, v).$$

Hence the triple (u, v, w) is also triple coincidence point of F and g. Thus we have

$$gu = gx, gv = gy \quad \text{and} \quad gw = gz.$$

Therefore

$$u = gu = F(u, v, w), v = gv = F(v, w, v) \quad \text{and} \quad w = F(w, u, v).$$

Thus (u, v, w) is common triple fixed point of F and g.

To prove the uniqueness, let (u^*, v^*, w^*) be any common triple fixed point of F and g. Then $u^* = gu = F(u^*, v^*, w^*), v^* = gv^* = F(v^*, w^*, v^*)$ and $w^* = F(w^*, u^*, v^*)$. Since the (u^*, v^*, w^*) is a triple coincidence point of F and g.

We have

$$gu^* = gx, gv^* = gy \quad \text{and} \quad gw^* = gz.$$

Thus

$$u^* = gu^* = gx = u, v^* = gv^* = gy = v \quad \text{and} \quad w^* = gw^* = gz = w.$$

Hence the common triple fixed point is unique.

This completes the proof.

As an immediate consequence of above theorem 3.1.1.1 we have the following corollary:

Corollary 3.1.1.3. Let $(X, d, \preceq)$ be an ordered cone metric space over a solid cone P. Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g - monotone property on X and there exists three elements $x_0, y_0, z_0 \in X$ with $gx_0 \preceq F(x_0, y_0, z_0), gy_0 \succeq F(y_0, z_0, y_0)$ and $gz_0 \preceq F(z_0, y_0, x_0)$. Suppose further that F, g satisfy that

$$d(F(x, y, z), F(u, v, w)) \preceq \alpha d(gx, gu) + \beta d(gy, gv) + \gamma d(gz, gw) + \delta d(F(x, y, z), gu),$$

for all $(x, y, z), (u, v, w) \in X$ with ($gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$), where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta < 1$. Further suppose

1. $F(X \times X \times X) \subseteq g(X)$;

2. $g(X)$ is a complete subspaces of X.

Also, suppose that if X has the following properties:

(i) $\{x_n\}$ is a non - decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in N$,

(ii) $\{y_n\}$ is a non - increasing sequence in X such that $y_n \to y$, then $y_n \succeq y$ for all $n \in N$,

(iii) $\{z_n\}$ is a non - decreasing sequence in X such that $z_n \to z$, then $z_n \preceq z$ for all $n \in N$,

then there exists $x, y, z \in X$ such that $g(x) = F(x, y, z), \quad g(y) = F(y, x, y) \quad \text{and} \quad g(z) = F(z, y, x)$, that is, F and g have tripled coincidence point in X.
Similarly corollary can be stated as a consequence of Theorem 3.1.1.1. Putting $g = i_X$ (the identity map) then we get the following corollary.

Corollary 3.1.1.4. Let $(X, d, \preceq)$ be complete ordered cone metric space over a solid cone P. Let
$F : X \times X \times X \to X$ be a mappings having the mixed monotone property on X and there exists three elements $x_0, y_0, z_0 \in X$ with $x_0 \preceq F(x_0, y_0, z_0), y_0 \succeq F(y_0, z_0, y_0)$ and $z_0 \preceq F(z_0, y_0, x_0)$. Suppose further that F, g satisfy

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &\preceq a_1 d(x,u) + a_2 d(F(x,y,z),x) + a_3 d(y,v) \\ &\quad + a_4 d(F(u,v,w),u) + a_5 d(F(x,y,z),u) + a_6 d(F(u,v,w),x), \end{aligned}$$

for all $(x, y, z), (u, v, w) \in X$ with $(x \preceq u, \quad y \succeq v \quad \text{and} \quad z \preceq w)$, where $a_i \geq 0$, for $i = 1, 2, ..., 7$ and $\sum_{i=1}^{6} a_i < 1$. Also suppose that if X has the following properties:

(i) $\{x_n\}$ is a non - decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in N$,

(ii) $\{y_n\}$ is a non - increasing sequence in X such that $y_n \to y$, then $y_n \succeq y$ for all $n \in N$,

(iii) $\{z_n\}$ is a non - decreasing sequence in X such that $z_n \to z$, then $z_n \preceq z$ for all $n \in N$,

then there exists $x, y, z \in X$ such that $x = F(x, y, z)$, $y = F(y, x, y)$ and $z = F(z, y, x)$, that is, F and g have tripled coincidence point in X.

Now, we furnish the following example to support our Theorem 3.1.1.1 and 3.1.1.2.

Example 3.1.1.5. Let $X = [0, 1]$ be taken with the standard order and with the cone metric given by $d(x, y) = (\mid x-y \mid, \alpha \mid x-y \mid)$ for fixed $\alpha > 0$. (Here $E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$ is a solid cone.) Consider the mappings $F : X \times X \times X \to X$ and $g : X \to X$ given by

$$F(x,y,z) = \begin{cases} \frac{1}{3}(x^2 - y^2 - z^2), & \text{if } x > y > z \\ 0 & \text{if } otherwise; \end{cases} \quad \text{and} \quad g(x) = x^2,$$

and the contractive condition taken with $a_1 = a_3 = \frac{1}{8}, a_2 = a_4 = \frac{1}{4}$ and $a_5 = a_6 = 0$. We will check that this condition is satisfied for all $x, y, z, u, v, w \in X$ with ($x \preceq u, y \succeq v$ and $z \preceq w$). The other conditions of Theorems are obviously satisfied. Consider the following possibilities.

Case1. $x > y > z$ and $u > v > w$. (and hence $u \geq x > y \geq v > w \geq z$). Then
$d(F(x, y, z), F(u, v, w)) = d(\frac{1}{3}(x^2 - y^2 - z^2), \frac{1}{3}(u^2 - v^2 - w^2)) = (L, \alpha L)$
where

$$L = \frac{1}{3}(u^2 - x^2 + v^2 - y^2 + z^2 - w^2),$$

and
$\frac{1}{8}d(gx, gu) + \frac{1}{4}d(F(x, y, z), gx) + \frac{1}{8}d(gy, gv) + \frac{1}{4}d(F(u, v, w), gu) = (D, \alpha D)$,
where $D = \frac{1}{8}(u^2 - x^2) + \frac{1}{12}(2x^2 + y^2 + z^2)) + \frac{1}{8}(y^2 - v^2) + \frac{1}{12}(2u^2 + v^2 + w^2))$ clearly $L \leq D$, Hence contraction condition (3.1.5) holds true.

Case2. $x > y > z$ and $u > w > v$ (and hence $u \geq x > y \geq w > v \geq z$). Then

$d(F(x,y,z), F(u,v,w)) = d(\frac{1}{3}(x^2 - y^2 - z^2), 0) = (L, \alpha L)$

where

$$L = \frac{1}{3}(x^2 - y^2 - z^2),$$

and

$\frac{1}{8}d(gx, gu) + \frac{1}{4}d(F(x,y,z), gx) + \frac{1}{8}d(gy, gv) + \frac{1}{4}d(0, gu) = (D, \alpha D),$

where

$$D = \frac{1}{8}(u^2 - x^2) + \frac{1}{12}(2x^2 + y^2 + z^2)) + \frac{1}{8}(y^2 - v^2) + \frac{1}{4}u^2$$

clearly $L \leq D$, Hence contraction condition (3.1.5) holds true.

Case3. $x > y > z$ and any other combination between u, v, w other than $u > v > w$. Then

$d(F(x,y,z), F(u,v,w)) = d(\frac{1}{3}(x^2 - y^2 - z^2), 0) = (L, \alpha L)$

where

$$L = \frac{1}{3}(x^2 - y^2 - z^2),$$

and

$\frac{1}{8}d(gx, gu) + \frac{1}{4}d(F(x,y,z), gx) + \frac{1}{8}d(gy, gv) + \frac{1}{4}d(0, gu) = (D, \alpha D),$

where

$$D = \frac{1}{8}(u^2 - x^2) + \frac{1}{12}(2x^2 + y^2 + z^2)) + \frac{1}{8}(y^2 - v^2) + \frac{1}{4}u^2$$

clearly $L \leq D$, Hence contraction condition (3.1.5) holds true.

Case4. $u > v > w$ and any other combination between x, y, z other than $x > y > z$. This case is treated analogously to the previous one.

Case5. Any other combination between x, y, z other than $x > y > z$ and also in between u, v, w other than $u > v > w$. Then

$d(F(x,y,z), F(u,v,w)) = d(0,0) = 0_E$ and the contractive condition is trivially satisfied.

Thus all the condition of Theorem 3.1.1.1 and 3.1.1.2 are satisfied. The mapping F and g have a unique common tripled fixed point $(0, 0, 0)$.

Chapter 4

SOME RESULTS IN FUZZY METRIC SPACES

4.1

We shall divided this chapter into three parts.

- In section - I, we have proved tripled coincidence point theorem and fixed point theorems for compatible maps.
- In section - II, we have proved theorem by using (f; g)-reciprocally continuity under Generalized (φ, ψ)-weak contractions maps, and
- Section - III, will be of Presič type contraction maps for obtaining common fixed points.

Now we are ready to discuss in detail about of the fuzzy metric spaces.

Zadeh([94]) of Univ. of California, Berkeley introduced the concept of fuzzy sets in 1965. There are many viewpoints of the notion of the metric space in fuzzy topology.We can divide them in two category:

(i). First category involves those results in which a fuzzy metric on a set X is treated as a map $d : X \times X \to \mathbb{R}^+$, where X represents the totality of all fuzzy points of a set

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and satisfy some axioms which are analogous to the ordinary metric axioms. In such an approach numerical distances are set up between fuzzy objects.

(ii). Second group studied those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.

Erceg ([104]), Kaleva and Seikkala ([107]) and Kramosil and Michalek ([75])discussed in length about fuzzy metric spaces. Grabiec ([105]) proved a **fixed point theorem** in **fuzzy metric space**. Subramanyam ([123]) generalized Grabiec's result for a pair of commuting maps in the pattern of Jungck 1976 ([52]). George and Veermani([11]) modified the concept of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space which have some important applications in quantum particle physics particularly in connection with both string and E - infinity theory. Also shown that every metric induces a fuzzy metric in Hausdroff topology.

Fuzzy Metric Spaces:

We recall the following definitions and lemmas for our results in fuzzy metric spaces.

Definition 4.1.1([94]) A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 4.1.2([76]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for every $a \in [0, 1]$;
4. $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 4.1.3([11]) A triplet $(X, M, *)$ is said to be a fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions;

for all $x, y, z \in X$ and $t, s > 0$;

(M_1) $M(x, y, t) > 0$,

(M_2) $M(x, y, t) = 1$ if and only if $x = y$,

(M_3) $M(x, y, t) = M(y, x, t)$,

(M_4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(M_5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In view of (M_1) and (M_2), it is worth pointing out that $0 < M(x, y, t) < 1$ for all $t > 0$, provided $x \neq y$. In view of Definition 4.1.3, George and Veermani ([11]) introduced the concept of Hausdorff topology on a fuzzy metric spaces and show that every metric space induces a fuzzy metric space.

In fact, we can fuzzify metric spaces into fuzzy metric spaces in a natural way as it can be shown by the following example. In other words, every metric induces a fuzzy metric.

Example 4.1.4 Let (X, d) be a metric space and define $a * b = ab$ for all $a, b \in [0, 1]$. Define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$, then $(X, M, *)$ is a fuzzy metric space, called standard fuzzy metric space induced by (X, d).

Definition 4.1.5([11]) Let $(X, M, *)$ be a fuzzy metric space, then

1. a sequence $\{x_n\}$ in X is said to be convergent to x if $lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$;

2. a sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if for a given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $M(x_n, x_m, t) > 1 - \epsilon$, for all $t > 0$ and $n, m \geq n_0$;

3. a fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

4.2 Section I

Sedghi, Altun and Shobe([152]) gave a coupled fixed point theorem for contractions in fuzzy metric spaces and Fang([83]) gave some common fixed point theorems under ϕ - contractions for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Many authors have proved fixed point theorems in (intuitionistic) fuzzy metric spaced or probabilistic metric spaces. Here we shall follow the proof of Hu([172]) for obtaining coincidence and fixed points.

We recall the following results:

Definition 4.2.1([117]) Let $\sup_{0<t<1} \triangle(t,t) = 1$. A t - norm $\triangle$ is said to be of H-type if the family of functions $\{\triangle^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\triangle^1(t) = t\triangle t, \quad \triangle^{m+1}(t) = t\triangle(\triangle^m(t)), \quad m = 1, 2, ..., t \quad \text{and} \quad \in [0, 1]. \qquad (4.2.1)$$

The t - norm $\triangle_M = \min$ is an example of t - norm of H - type, but there are some other t - norms $\triangle$ of H - type ([117]).

Obviously, $\triangle$ is a H - type t norm if and only if for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\triangle^m(t) > 1 - \lambda$ for all $m \in N$, when $t > 1 - \delta$.

Remark 4.2.2(see [105]).

(i) For all $x, y \in X, M(x, y, .)$ is non - decreasing.

(ii) It is easy to prove that if $x_n \to x, y_n \to y, t_n \to t$, then

$$lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t). \tag{4.2.2}$$

(iii) In a fuzzy metric space $(X, M, *)$, whenever $M(x, y, t) > 1 - r$ for $x, y \in X, t > 0, 0 < r < 1$, we can find a $t_0, 0 < t_0 < t$ such that $M(x, y, t_0) > 1 - r$.

(iv) For any $r_1 > r_2$, we can find an r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find r_5 such that $r_3 * r_5 \geq r_4$, where $r_1, r_2, r_3, r_4, r_5 \in (0, 1))$.

Definition 4.2.3([152]) Let $(X, M, *)$ be a fuzzy metric space and M is said to satisfy the n - property on $X \times X \times (0, \infty)$ if

$$lim_{n\to\infty} [M(x, y, k^n t)]^{n^p} = 1, \tag{4.2.3}$$

whenever $x, y \in X, k > 1$ and $p > 0$.

Lemma 4.2.4 Let $(X, M, *)$ be a fuzzy metric space. If M satisfies the n - property, then

$$lim_{t\to+\infty} M(x, y, t) = 1, \qquad \forall x, y \in X. \tag{4.2.4}$$

Proof. If not, since $M(x, y, .)$ is non - decreasing and $0 \leq M(x, y, .) \leq 1$, there exists $x_0, y_0 \in X$ such that $lim_{t\to+\infty} M(x_0, y_0, t) = \lambda < 1$, then for $k > 1, k^n t \to +\infty$ as $n \to \infty$ and $t > 0$. We get $lim_{n\to\infty}[(M(x_0, y_0, k^n t))^{n^p}] = 0$, which is a contraction.

Remark 4.2.5 Condition (4.2.4) cannot guarantee the n - property. For this see the following example.

Example 4.2.6([172]) Let (X, d) be an ordinary metric space, $a * b \leq ab$ for all $a, b \in [0, 1]$, and ψ be define as following:

$$\psi(t) = \begin{cases} \alpha\sqrt{t}, & \text{if} \quad 0 < t \leq 4, \\ 1 - \frac{1}{Int}, & \text{if} \quad t > 4, \end{cases} \tag{4.2.5}$$

where $\alpha = (\frac{1}{2})(1 - \frac{1}{In4})$. Then $\psi(t)$ is continuous and increasing in $(0, \infty)$, $\psi(t) \in (0, 1)$ and $\lim_{t \to +\infty} \psi(t) = 1$. Let

$$M(x, y, t) = [\psi(t)]^{d(x,y)}, \qquad \forall x, y \in X, t > 0, \tag{4.2.6}$$

then $(X, M, *)$ is a fuzzy metric space and

$$\lim_{t \to +\infty} M(x, y, t) = \lim_{t \to +\infty} [\psi(t)]^{d(x,y)} = 1, \qquad \forall \quad x, y \in X, t > 0, \tag{4.2.7}$$

But for any $x \neq y, p = 1, k > 1, t > 0$,

$$\lim_{n \to \infty} [M(x, y, k^n t)]^{n^p} = \lim_{n \to \infty} [(\psi(k^n t))]^{d(x,y).n^p} = \lim_{n \to \infty} [1 - \frac{1}{In(k^n t)}]^{n.d(x,y)} = e^{\frac{-d(x,y)}{Ink}} \neq 1. \tag{4.2.8}$$

Now, we recall the following:

$\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, \infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

$(\phi_{(1)})$ ϕ is non decreasing;

$(\phi_{(2)})$ ϕ is upper semi - continuous from the right;

$(\phi_{(3)})$ $\Sigma_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$ where $\phi^{n+1}(t) = \phi(\phi^n(t)), n \in N$.

It is easy to prove that, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Lemma 4.2.7([83]) Let $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous t -

norm of H - type. If there exists $\phi \in \Phi$ such that

$$M(x, y, \phi(t)) \geq M(x, y, t), \quad \text{for all } t > 0, \tag{4.2.9}$$

then $x = y$.

In this context, we slightly modify the concept of tripled fixed point introduced by Brinde and Borcut([157]) in POMS into fuzzy metric space we also introduce the following notion of tripled fixed point and tripled coincidence point.

Definition 4.2.8 Let $(X, M, *)$ be a nonempty fuzzy metric space. An element $(x, y, z) \in X \times X \times X$ is said to be tripled fixed point of the map $F : X \times X \times X \to X$ if $F(x, y, z) = x$, $F(y, x, z) = y$ and $F(z, y, x) = z$.

Definition 4.2.9 Let $(X, M, *)$ be a nonempty fuzzy metric space. An element $(x, y, z) \in X \times X \times X$ is said to be tripled coincidence point of the maps $F : X \times X \times X \to X$ and $g : X \to X$ if $F(x, y, z) = g(x)$, $F(y, x, z) = g(y)$ and $F(z, y, x) = g(z)$.

Now we define the concept of compatibility for a pair of maps $F : X \times X \times X \to X$ and $g : X \to X$.
The notion introduce by us is analogous to the notion of Hu([172]) in the setting of Fuzzy Metric Spaces.

Definition 4.2.10 The maps $F : X \times X \times X \to X$ and $g : X \to X$ are said to be compatible pair of maps if

$$\lim_{n\to\infty} M(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n)), t) = 1,$$

$$\lim_{n\to\infty} M(g(F(y_n, x_n, z_n)), F(g(y_n), g(x_n), g(z_n)), t) = 1,$$

and

$$\lim_{n\to\infty} M(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n)), t) = 1,$$

whenever $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = x, \lim_{n\to\infty} F(y_n, x_n, z_n) = \lim_{n\to\infty} g(y_n) = y$$

$$\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = z,$$

for all $x, y, z \in X$ and $t > 0$.

Definition 4.2.11 The mappings $F : X \times X \times X \to X$ and $g : X \to X$ are said to be commutative if

$$g(F(x, y, z)) = F(gx, gy, gz),$$

for all $x, y \in X$.

Remark 4.2.12 If F and g are commutative pair of maps then they are compatible pair of maps.

4.2.1 Main Results

Now, we prove some results of tripled coincidence point theorem under weaker conditions, which generalizes the result of Hu([172]).

Throughout this section:

$$[M(x, y, t)]^n = \underbrace{M(x, y, t) * M(x, y, t) * ... * M(x, y, t),}_{n-times} \tag{4.2.10}$$

for all $n \in \mathbb{N}$.

Theorem 4.2.1.1 Let $(X, M, *)$ be a complete FM - Space, where $*$ is a continuous t

- norm of H - type satisfying (4.2.4). Let $F : X \times X \times X \to X$ and $g : X \to X$ be mappings such that there exists $\phi \in \Phi$ satisfying

$$M(F(x,y,z), F(u.v,w), \phi(t)) \geq M(g(x), g(u), t) * M(g(y), g(v), t) * M(g(z), g(w), t), \tag{4.2.11}$$

for all $x, y, z, u, v, w \in X, \quad t > 0$. Suppose that

(a) $F(X \times X \times X) \subseteq g(X)$;

(b) g is continuous;

(c) F and g are compatible.

Then there exists $x, y, z \in X$ such that $g(x) = F(x,y,z), \quad g(y) = F(y,x,z), \quad g(z) = F(z,y,x)$ i.e., F and g have a triple coincidence point in X.

Proof. Let $x_0, y_0, z_0 \in X$ be arbitrary points in X. Since $F(X \times X \times X) \subseteq g(X)$, then there exists $x_1, y_1, z_1 \in X$ such that

$g(x_1) = F(x_0, y_0, z_0), g(y_1) = F(y_0, x_0, z_0)$, and $g(z_1) = F(z_0, y_0, x_0)$.

Continuing in this way, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n, z_n), g(y_{n+1}) = F(y_n, x_n, z_n) \quad \text{and} \quad g(z_{n+1}) = F(z_n, y_n, x_n), \tag{4.2.12}$$

for all n ≥ 0.

We shall divided the proof in two steps:

Step I: First we show that $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Since $*$ is a t - norm of H - type, then for any $\lambda > 0$, there exists a $\mu > 0$ such that

$$\underbrace{(1-\mu) * (1-\mu) * ... * (1-\mu)}_{k-times} \geq 1 - \lambda, \tag{4.2.13}$$

for all $k \in N$.

Since $M(x,y,.)$ is continuous and $lim_{t\to+\infty}M(x,y,t) = 1$ for all $x, y \in X$, there exists $t_0 > 0$ such that

$$\begin{cases} M(gx_0, gx_1, t_0) \geq 1-\mu, \\ M(gy_0, gy_1, t_0) \geq 1-\mu, \\ M(gz_0, gz_1, t_0) \geq 1-\mu. \end{cases} \tag{4.2.14}$$

On the other hand, since $\phi \in \Phi$, by condition $(\phi_{(3)})$ we have $\sum_{n=1}^{\infty} \phi^n(t_0) < \infty$. Then for any $t > 0, \exists \quad n_0 \in \mathbb{N}$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0) \tag{4.2.15}$$

From condition (4.2.11), we have

$$M(gx_1, gx_2, \phi(t_0)) = M(F(x_0, y_0, z_0), F(x_1, y_1, z_1), \phi(t_0))$$

$$\geq M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0) * M(gz_0, gz_1, t_0).$$

$$M(gy_1, gy_2, \phi(t_0)) = M(F(y_0, x_0, z_0), F(y_1, x_1, z_1), \phi(t_0))$$

$$\geq M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0) * M(gz_0, gz_1, t_0).$$

and

$$M(gz_1, gz_2, \phi(t_0)) = M(F(z_0, y_0, x_0), F(z_1, y_1, x_1), \phi(t_0))$$

$$\geq M(gz_0, gz_1, t_0) * M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0)$$

Similarly, we can also get

$$M(gx_2, gx_3, \phi^2(t_0)) = M(F(x_1, y_1, z_1), F(x_2, y_2, z_2), \phi^2(t_0))$$

$$\geq M(gx_1, gx_2, \phi(t_0)) * M(gy_1, gy_2, \phi(t_0)) * M(gz_1, gz_2, \phi(t_0))$$

$$= M(F(x_0, y_0, z_0), F(x_1, y_1, z_1), \phi(t_0))$$

$$*M(F(y_0,x_0,z_0),F(y_1,x_1,z_1),\phi(t_0))$$

$$*M(F(z_0,y_0,x_0),F(z_1,y_1,x_1),\phi(t_0))$$

$$\geq M(gx_0,gx_1,t_0)*M(gy_0,gy_1,t_0)*M(gz_0,gz_1,t_0)$$

$$*M(gy_0,gy_1,t_0)*M(gx_0,gx_1,t_0)*M(gz_0,gz_1,t_0)$$

$$*M(gz_0,gz_1,t_0)*M(gy_0,gy_1,t_0)*M(gx_0,gx_1,t_0)$$

$$=[M(gx_0,gx_1,t_0)]^3*[M(gy_0,gy_1,t_0)]^3*[M(gz_0,gz_1,t_0)]^3.$$

$$M(gy_2,gy_3,\phi^2(t_0))=M(F(y_1,x_1,z_1),F(y_2,x_2,z_2),\phi^2(t_0))$$

$$\geq M(gy_1,gy_2,\phi(t_0))*M(gx_1,gx_2,\phi(t_0))*M(gz_1,gz_2,\phi(t_0))$$

$$=M(F(y_0,x_0,z_0),F(y_1,x_1,z_1),\phi(t_0))$$

$$*M(F(x_0,y_0,z_0),F(x_1,y_1,z_1),\phi(t_0))$$

$$*M(F(z_0,y_0,x_0),F(z_1,y_1,x_1),\phi(t_0))$$

$$\geq M(gy_0,gy_1,t_0)*M(gx_0,gx_1,t_0)*M(gz_0,gz_1,t_0)$$

$$*M(gx_0,gx_1,t_0)*M(gy_0,gy_1,t_0)*M(gz_0,gz_1,t_0)$$

$$*M(gz_0,gz_1,t_0)*M(gy_0,gy_1,t_0)*M(gx_0,gx_1,t_0)$$

$$=[M(gy_0,gy_1,t_0)]^3*[M(gx_0,gx_1,t_0)]^3*[M(gz_0,gz_1,t_0)]^3.$$

$$M(gz_2,gz_3,\phi^2(t_0))\geq[M(gz_0,gz_1,t_0)]^3*[M(gy_0,gy_1,t_0)]^3*[M(gx_0,gx_1,t_0)]^3.$$

Continuing in this way, we get

$$M(gx_n,gx_{n+1},\phi^n(t_0))\geq[M(gx_0,gx_1,t_0)]^{3^{n-1}}*[M(gy_0,gy_1,t_0)]^{3^{n-1}}*[M(gz_0,gz_1,t_0)]^{3^{n-1}},$$

$$M(gy_n, gy_{n+1}, \phi^n(t_0)) \geq [M(gy_0, gy_1, t_0)]^{3^{n-1}} * [M(gx_0, gx_1, t_0)]^{3^{n-1}} * [M(gz_0, gz_1, t_0)]^{3^{n-1}},$$

$$M(gz_n, gz_{n+1}, \phi^n(t_0)) \geq [M(gz_0, gz_1, t_0)]^{3^{n-1}} * [M(gy_0, gy_1, t_0)]^{3^{n-1}} * [M(gx_0, gx_1, t_0)]^{3^{n-1}},$$

So, from (4.2.14) and (4.2.15), for $m > n \geq n_0$, we have

$$\begin{aligned}
M(gx_n, gx_m, t_0) &\geq M(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)) \\
&\geq M(gx_n, gx_m, \sum_{k=n}^{m-1} (t_0)) \\
&\geq M(gx_n, gx_{n+1}, \phi^n(t_0)) * M(gx_{n+1}, gx_{n+2}, \phi^{n+1}(t_0)) \\
&\quad * \cdots * M(gx_{m-1}, gx_m, \phi^{m-1}(t_0)) \\
&\geq [(M(gy_0, gy_1, t_0))^3]^{n-1} * [(M(gx_0, gx_1, t_0)^{3^{n-1}} * [M(gz_0, gz_1, t_0)]^{3^{n-1}} \\
&\quad *[M(gy_0, gy_1, t_0)]^{3^n} * [M(gx_0, gx_1, t_0)]^{3^n} * [M(gz_0, gz_1, t_0)]^{3^n} \\
&\quad * \cdots *[M(gy_0, gy_1, t_0)]^{3^{m-2}} *[M(gx_0, gx_1, t_0)]^{3^{m-2}} *[M(gz_0, gz_1, t_0)]^{3^{m-2}} \\
&= [(M(gy_0, gy_1, t_0))^{\frac{1}{2}}]^{3^{n-1}(3^{m-n}-1)} * [(M(gx_0, gx_1, t_0)^{\frac{1}{2}}]^{3^{n-1}(3^{m-n}-1)} \\
&\quad *[(M(gz_0, gz_1, t_0))^{\frac{1}{2}}]^{3^{n-1}(3^{m-n}-1)} \\
&\geq (1-\mu) * (1-\mu) * ... * (1-\mu) \\
&\geq 1-\lambda
\end{aligned}$$

which implies that

$$M(gx_n, gx_m, t) > 1 - \lambda, \tag{4.2.16}$$

for all $m, n \in N$ with $m > n \geq n_0$ and $t > 0$. So $\{g(x_n)\}$ is a cauchy sequence. Similarly, we have $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences.

Step II: Here we shall prove that g and F have a tripled coincidence point. Since X is complete, there exists $x, y, z \in X$ such that

$$lim_{n\to\infty}F(x_n, y_n, z_n) = lim_{n\to\infty}g(x_n) = x,$$

$$lim_{n\to\infty}F(y_n, x_n, z_n) = lim_{n\to\infty}g(y_n) = y,$$

$$lim_{n\to\infty}F(z_n, y_n, x_n) = lim_{n\to\infty}g(z_n) = z,$$

Suppose F and g are compatible, so we have

$$lim_{n\to\infty}M(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), t) = 1,$$

$$lim_{n\to\infty}M(gF(y_n, x_n, z_n), F(gy_n, gx_n, gz_n), t) = 1,$$

$$lim_{n\to\infty}M(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n), t) = 1.$$

$\forall \quad t > 0,$

Now we prove that $g(x) = F(x, y, z), \quad g(y) = F(y, x, z)$ and $g(z) = F(z, y, x)$

Now by condition (4.2.11) and (4.2.12), we have

$$M(gx, F(x, y, z), \phi(t)) \geq M(ggx_{n+1}, F(x, y, z), \phi(k_1t)) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1t))$$

$$= M(gF(x_n, y_n, z_n), F(x, y, z), \phi(k_1t)) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1t))$$

$$\geq M(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), \phi(k_1t) - \phi(k_2t))$$

$$*M(F(gx_n, gy_n, gz_n), F(x, y, z), \phi(k_2t))$$

$$*M(gx, ggx_{n+1}, \phi(t) - \phi(k_1t))$$

$$\geq M(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n), \phi(k_1t) - \phi(k_2t))$$

$$*M(ggx_n, gx, k_2t) * M(ggy_n, gy, k_2t)$$

$$*M(ggz_n, gz, k_2t) * M(gx, ggx_{n+1}, \phi(t) - \phi(k_1t)),$$

$\forall \quad 0 < k_2 < k_1 < 1$. Since g is continuous and $\{g, F\}$ are compatible pair and letting $n \to \infty$, then we have $M(gx, F(x, y, z), \phi(t)) \geq 1$, implies $gx = F(x, y, z)$.

Similarly we can get $gy = F(y, x, z)$ and $gz = F(z, y, x)$.

Hence we have shown that F and g have triple coincidence point.

As an immediate consequence of the above theorem, we have the following corollaries:

Corollary 4.2.1.2 Let $(X, M, *)$ be a complete FM - Space, where $*$ is a continuous t - norm of H - type satisfying (4.2.4). Let $F : X \times X \times X \to X$ and there exists $\phi \in \Phi$ such that

$$M(F(x,y,z), F(u.v,w), \phi(t)) \geq M(x,u,t) * M(y,v,t) * M(z,w,t), \tag{4.2.17}$$

for all $x, y, z, u, v, w \in X, t > 0$. Then there exits $x, y, z \in X$ such that $x = F(x,y,z), y = F(y,x,z), z = F(z,y,x)$ i.e., F have a common tripled fixed point in X.

Corollary 4.2.1.3([152]) Let $a * b \geq ab$ for all $a, b \in [0,1]$ and $(X, M, *)$ be a complete FM - Space such that M has n - property. Let $F : X \times X \times X \to X$ and $g : X \to X$ be two function such that

$$M(F(x,y,z), F(u.v,w), kt) \geq M(gx, gu, t) * M(gy, gv, t) * M(gz, gw, t), \tag{4.2.18}$$

for all $x, y, z, u, v, w \in X$, where $0 < k < 1, \quad F(X \times X \times X) \subset g(X)$ and g is continuous and commutes with F.

Then there exists triple coincidence point $(x, y, z) \in X$ such that $g(x) = F(x,y,z), g(y) = F(y,x,z), g(z) = F(z,y,x)$.

Now we are ready to give an example which validate our main theorem:

Example 4.2.1.4 Let $X = [-2,2], a * b = ab$ for all $a, b \in [0,1]$. ψ is defined as (4.2.5). Let

$$M(x,y,t) = [\psi(t)]^{|x-y|},$$

for all $x, y \in [0,1]$. Then $(X, M, *)$ is a complete FM - space.

Let $\phi(t) = \frac{t}{2},\ g(x) = x$ and $F : X \times X \times X \to X$ be defined as

$$F(x, y, z) = \frac{x + y + z}{4} \qquad \forall x, y \in X.$$

Thus F satisfies all the condition of Theorem 4.2.1.1, and the points $(0,0,0)$ are tripled coincidence points of g and F.

It is easy to see that $F(X \times X \times X) = [-\frac{3}{2}, \frac{3}{2}]$,

$$M(F(x, y, z), F(u, v, w), \phi(t)) = [\psi(\phi(t))]^{\frac{|x-u+y-v+z-w|}{4}}, \tag{4.2.19}$$

For all $t > 0$ and $x, y \in [-2, 2]$. The condition (4.2.17) is equivalent to

$$[\psi(\frac{t}{2})]^{\frac{|x-u+y-v+z-w|}{4}} \geq [\psi(t)]^{|x-u|} \cdot [\psi(t)]^{|y-v|} \cdot [\psi(t)]^{|z-w|} \tag{4.2.20}$$

Since $\psi \in (0, 1)$, we can get

$$[\psi(\frac{t}{2})]^{\frac{|x-u+y-v+z-w|}{4}} \geq [\psi(\frac{t}{2})]^{\frac{|x-u|}{4}} \cdot [\psi(\frac{t}{2})]^{\frac{|y-v|}{4}} \cdot [\psi(\frac{t}{2})]^{\frac{|z-w|}{4}} \tag{4.2.21}$$

From (4.2.20), we only need to verify the following:

$$[\psi(\frac{t}{2})]^{\frac{|x-u|}{4}} \geq [\psi(t)]^{|x-u|} \tag{4.2.22}$$

that is,

$$\psi(\frac{t}{2}) \geq [\psi(t)]^4 \forall t > 0. \tag{4.2.23}$$

Now, we consider the following cases.

Case 1: $(0 < t \leq 4)$. Then (4.2.23) is equivalent to

$$\alpha\sqrt{\frac{t}{2}} \geq (\alpha\sqrt{(}t))^4, \tag{4.2.24}$$

and it can be easily to verify.

Case 2: $(t \geq 8)$. Then (4.2.23) is equivalent to

$$1 - \frac{1}{In\frac{t}{2}} \geq (1 - \frac{1}{Int})^4 \tag{4.2.25}$$

equivalent is

$$4In^3t \cdot In\frac{t}{2} + 4Int \cdot In\frac{t}{2} \geq In^4t + 6In^2t \cdot In\frac{t}{2} + In\frac{t}{2} \tag{4.2.26}$$

since

$$In^4t - 4In^3t \cdot In\frac{t}{2} - 4Int \cdot In\frac{t}{2} + 6In^2t \cdot In\frac{t}{2} + In^4\frac{t}{2} - In^4\frac{t}{2} + In\frac{t}{2} \leq 0, \tag{4.2.27}$$

hods for all $t \geq 8$. So (4.2.23) holds for $t \geq 8$.

Case 3: $(4 < t < 8)$. Then (4.2.23) is equivalent to

$$\alpha\sqrt{\frac{t}{2}} \geq (1 - \frac{1}{Int})^4 \tag{4.2.28}$$

Let $t = e^x$, we only need to verify

$$\frac{\alpha}{\sqrt{2}}e^{\frac{x}{2}} - (1 - \frac{1}{x})^4 \geq 0, \tag{4.2.29}$$

for all x that $2In2 < x < 3In2$. We can verify it holds.

Hence it is verified that the functions F, g, ϕ satisfy all the conditions of Theorem 4.2.1.1

and $(0,0,0)$ are the triple coincidence point of F and g in X.

4.3 Section II

In this section, we recall the following definitions.

In 1984, Khan, Swaleh and Sessa ([116]) have introduced the concept of an altering distance function in metric spaces in the following manner:

Definition 4.3.1 A function $\varphi : [0,\infty) \to [0,\infty)$ is an altering distance function, if $\varphi(t)$ is monotone non-decreasing, continuous and $\varphi(t) = 0$ iff $t = 0$.

Now we introduce the notion of (f,g)-reciprocal continuity in Fuzzy Metric Space:

Definition 4.3.2 Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be (f,g)-reciprocally continuous iff $\lim_{n\to\infty} ff(x_n) = fu$ and $\lim_{n\to\infty} gg(x_n) = gu$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$.

In 1994, Mishra, Sharma and Singh [148] have introduced the concept of compatible maps in fuzzy metric spaces which follows:

Definition 4.3.3 Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be compatible if $\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and for all $t > 0$.

In 1996, Pathak, Cho, Chang and Kang([71]) introduced the concept of compatible maps of type (P) in metric spaces and given its relationship with compatible maps and compatible maps of type(A) in metric spaces introduced by Jungck [53] and Jungck, Murthy and Cho [56], respectively.

According to this notion, we give the following definition:

Definition 4.3.4 Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be weakly compatible of type (P) iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and $\lim_{n\to\infty} M(ff(x_n), gg(x_n), t) = 1$, for all $t > 0$.

Here we observe the following result:

Proposition 4.3.5 Let f and g be self maps of a fuzzy metric space $(X, M, *)$ and let the pair be (f, g)-reciprocally continuous maps. Then f and g have a coincidence point iff they are weakly compatible maps of type (P).

Proof. (If part) Suppose that f and g have a coincidence point, say u. If we consider a sequence $\{x_n\}$ in X such that $x_n = u$ for all $n \geq 0$, then we have

$$\lim_{n\to\infty} f(x_n) = fu = z = gu = \lim_{n\to\infty} g(x_n),$$

for some $z \in X$.

Since f and g are (f, g)-reciprocally continuous maps, then $\lim_{n\to\infty} ff(x_n) = fu$ and $\lim_{n\to\infty} gg(x_n) = gu$. This implies that $\lim_{n\to\infty} M(ff(x_n), gg(x_n), t) = M(fu, gu, t) = 1$ and therefore, f and g are weakly compatible maps of type (P).

(Only if part) Now, assume that f and g are weakly compatible maps of type (P). Corresponding to a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and using (M_4) of Definition 4.1.3, we can write

$$M(fu, gu, t) \geq M(fu, ff(x_n), \tfrac{t}{3}) * M(ff(x_n), gg(x_n), \tfrac{t}{3}) * M(gg(x_n), gu, \tfrac{t}{3}).$$

Taking $n \to \infty$, we get

$M(fu, gu, t) \geq \lim_{n\to\infty} M(fu, ff(x_n), \frac{t}{3}) * \lim_{n\to\infty} M(ff(x_n), ggx_n, \frac{t}{3})$

$*\lim_{n\to\infty} M(gg(x_n), gu, \frac{t}{3})$,

which implies, since f and g are (f,g)-reciprocally continuous and weakly compatible maps of type (P), that $M(fu, gu, t) = 1$. Therefore, f and g have a coincidence point.

Gopal and Imdad ([38]) studied the concept of sub-compatible maps in fuzzy metric spaces. Subsequently, Murthy and Tas discussed and utilized it in ([126]). We recall that this concept was initially introduced by Bouhadjera and Thobie ([63]) in metric spaces to weaken the notion of occasionally weakly compatible maps ([113]) and weak compatible maps [57]. On this topic, we ask the reader to see [36, 39, 149].

Definition 4.3.6 Let f and g be self-maps of a fuzzy metric space $(X, M, *)$. The maps f and g are said to be sub-compatible iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and

$$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1,$$

for all $t > 0$.

Now, we give some illustrative examples of sub-compatible and (f,g)-reciprocally continuous maps with coincidence point and without coincidence point.

Example 4.3.7 Let $X = [0,\infty)$ and $M(x,y,t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} x^2, & \text{if } x \in [0,1) \\ 2x, & \text{if } x \in [1,\infty) \end{cases} \text{ and } g(x) = \begin{cases} x, & \text{if } x \in [0,1) \\ x+1, & \text{if } x \in [1,\infty) \end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $\{x_n\} = 1 - \frac{1}{n+1}$ for all $n \geq 0$. Then, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} fg(x_n) = \lim_{n\to\infty} gf(x_n) = 1$$

and

$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1.$

Therefore, the maps f and g are sub-compatible.

On the other hand the maps f and g are not compatible. In fact, let $\{x_n\}$ be a sequence in X such that $\{x_n\} = 1 + \frac{1}{n+1}$, for all $n \geq 0$. Then, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = 2$, $\lim_{n\to\infty} fg(x_n) = 4$, $\lim_{n\to\infty} gf(x_n) = 3$ and so

$$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = \frac{t}{t+1} \neq 1.$$

Example 4.3.8 Let $X = \mathbb{R}$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in (-\infty, 1) \\ x, & \text{if } x \in [1, \infty) \end{cases} \quad \text{and } g(x) = \begin{cases} x+1, & \text{if } x \in (-\infty, 1) \\ 2x-1, & \text{if } x \in [1, \infty) \end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $x_n = 1 + \frac{1}{n}$, for all $n \geq 1$. Then, we have

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} fg(x_n) = \lim_{n\to\infty} gf(x_n) = 1$

and

$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1.$

Therefore, the maps f and g are sub-compatible.

On the other hand, the maps f and g are not compatible. In fact, let $\{x_n\}$ be a sequence in X such that $\{x_n\} = \frac{1}{n} - 2$, for all $n \geq 1$. Then, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = -1$, $\lim_{n\to\infty} fg(x_n) = -\frac{1}{2}$, $\lim_{n\to\infty} gf(x_n) = 0$ and so

$$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = \frac{t}{t+1/2} \neq 1.$$

It is easy to show that f and g are (f, g)-reciprocally continuous maps with coincidence point $x = 1$ (that is also a common fixed point of the pair (f, g)) and this implies that f and g are weakly compatible maps of type (P).

Example 4.3.9 Let $X = [0, \infty)$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X$ and $t > 0$. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} 1 + x, & \text{if } x \in [0, 1); \\ x, & \text{if } x \in [1, \infty); \end{cases} \quad \text{and } g(x) = \begin{cases} 1 - x, & \text{if } x \in [0, 1]; \\ 2x - 1, & \text{if } x \in (1, \infty). \end{cases}.$$

Notice that f and g are discontinuous. Let $\{x_n\}$ be a sequence in X such that $\{x_n\} = 1 + \frac{1}{n+1}$, for all $n \geq 0$. Then, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} fg(x_n) = \lim_{n\to\infty} gf(x_n) = 1,$$

and

$$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1.$$

Therefore, the maps f and g are sub-compatible.

On the other hand, the maps f and g are not compatible. In fact, let $\{x_n\}$ be a sequence in X such that $\{x_n\} = \frac{1}{2n}$, for all $n \geq 1$. Then, we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = 1$, $\lim_{n\to\infty} fg(x_n) = 2$, $\lim_{n\to\infty} gf(x_n) = 0$ and so

$$\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = \frac{t}{t+2} \neq 1.$$

It is easy to show that f and g are (f, g)-reciprocally continuous maps without a coincidence point. Therefore, f and g are not weakly compatible maps of type (P).

4.3.1 Main Results

Before entering into our theorem and its proof, We need the following definitions given by Gopal and Imdad([9]) in fuzzy metric spaces.

Definition 4.3.1.1 Let $(X, M, *)$ be a fuzzy metric space and $f, g : X \to X$ be given maps. The map g is called a generalized (φ, ψ)-weak contraction with respect to f if there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ such that

$$\varphi\left(\frac{1}{M(gx, gy, t)} - 1\right) \leq \varphi\left(\frac{1}{m(f, g)} - 1\right) - \psi\left(\frac{1}{m(f, g)} - 1\right) \tag{4.3.1}$$

holds for all $x, y \in X$ and each $t > 0$ with

$$m(f, g) = \min\{M(fx, fy, t), M(gx, fx, t), M(gy, fy, t)\}.$$

If $f = I_X$, where I_X is the identity map, then g is called a generalized (φ, ψ)-weak contraction.

Theorem 4.3.1.2 Let $f, g : X \to X$ be (f, g)-reciprocally continuous self-maps of a fuzzy metric space $(X, M, *)$ such that

1. $g(X) \subseteq f(X)$,
2. one of $f(X)$ and $g(X)$ is a G-complete subset of X,
3. g is a generalized (φ, ψ)-weak contraction with respect to f.

If f and g are sub-compatible maps and ψ is a continuous function, then f and g have a unique common fixed point in X, that is, there exists $u \in X$ such that $u = fu = gu$.
Proof. Let x_0 be an arbitrary point in X. Since $g(X) \subseteq f(X)$, we can define, for each

$n \geq 0$, a sequence of points $x_0, x_1, x_2, \ldots, x_n, \ldots$, such that x_{n+1} is in the pre-image under f of $\{g(x_n)\}$, that is, $g(x_0) = f(x_1), g(x_1) = f(x_2), \ldots, g(x_n) = f(x_{n+1}), \ldots$ for every $n \geq 0$. Moreover, we can define a sequence y_n in X by

$$y_n = g(x_n) = f(x_{n+1}), \text{ for all } \quad n \geq 0. \tag{4.3.2}$$

Suppose that $y_n = y_{n+1}$ for some n. Then by condition (4.3.1) we have easily $y_{n+1} = y_{n+2}$ and so $y_m = y_n$ for every $m > n$. Thus the sequence $\{y_n\}$ is Cauchy sequence. Assume that $y_{n+1} \neq y_n$, for all $n \geq 0$. Then, for $x = x_{n+1}$ and $y = x_n$, we have

$$\begin{aligned} m(f,g) &= \min\{M(f(x_{n+1}), f(x_n), t), M(g(x_{n+1}), f(x_{n+1}), t), M(g(x_n), f(x_n), t)\} \\ &= \min\{M(y_n, y_{n-1}, t), M(y_{n+1}, y_n, t), M(y_n, y_{n-1}, t)\}. \end{aligned}$$

Now, if $m(f,g) = M(y_{n+1}, y_n, t)$, we obtain

$$\begin{aligned} \varphi\left(\frac{1}{M(g(x_{n+1}), g(x_n), t)} - 1\right) &= \varphi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) - \psi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) \end{aligned}$$

which implies that $M(y_{n+1}, y_n, t) = 1$, a contradiction as $y_{n+1} \neq y_n$ for all n. Then, we must have $m(f,g) = M(y_n, y_{n-1}, t)$ and hence

$$\begin{aligned} \varphi\left(\frac{1}{M(y_{n+1}, y_n, t)} - 1\right) &\leq \varphi\left(\frac{1}{M(y_{n-1}, y_n, t)} - 1\right) - \psi\left(\frac{1}{M(y_{n-1}, y_n, t)} - 1\right) \\ &< \varphi\left(\frac{1}{M(y_n, y_{n-1}, t)} - 1\right). \end{aligned}$$

Consequently, considering that the φ function is non-decreasing, we have that

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t), \text{ for all } n,$$

and hence $\{M(y_{n-1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $(0, 1]$.

Let $S(t) = \lim_{n\to\infty} M(y_{n-1}, y_n, t)$. Now, we show that $S(t) = 1$, for all $t > 0$. If not, there exists $t > 0$ such that $S(t) < 1$. Then from the above inequality, on taking $n \to \infty$, we obtain

$$\varphi\left(\frac{1}{S(t)} - 1\right) \leq \varphi\left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right),$$

and arrives at a contradiction. Therefore $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$.

Now, for each positive integer p, we write

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, \frac{t}{p}) * M(y_{n+1}, y_{n+2}, \frac{t}{p}) * \cdots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}).$$

It follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1,$$

and hence $\{y_n\}$ is a G-cauchy sequence.

Since $f(X)$ is G-complete, then there exists $u \in f(X)$ such that $y_n \to u$ as $n \to \infty$.

Clearly,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_{n+1}) = u.$$

Now, (f, g)-reciprocal continuity of f and g implies that $ff(x_n) \to fu$ and $gg(x_n) \to gu$, as $n \to \infty$. From $ff(x_n) \to fu$, by construction of the sequence (4.3.2), we have $fg(x_n) = ff(x_{n+1}) \to fu$, as $n \to \infty$. On the other hand, sub-compatibility of f and g yields $\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1$ which implies that $gf(x_n) \to fu$. Now, taking $x = u$ and $y = f(x_n)$, we get

$$m(f, g) = \min\{M(fu, ff(x_n), t), M(gu, fu, t), M(gf(x_n), ff(x_n), t)\}$$

$$\to M(gu, fu, t),$$

as $n \to \infty$. Now, by

$$\varphi\left(\tfrac{1}{M(gu, gf(x_n), t)} - 1\right) \leq \varphi\left(\tfrac{1}{m(f,g)} - 1\right) - \psi\left(\tfrac{1}{m(f,g)} - 1\right),$$

on taking $n \to \infty$, we get $M(gu, fu, t) = 1$, which implies that $gu = fu$. It means that u is a coincidence point of f and g.

Now, we shall show that u is also a common fixed point of f and g. For this we assert that $gu = u$, and so $gu = u = fu$. On the other hand, if $gu \neq u$, then taking $x = u$ and $y = x_n$, we have

$$m(f, g) = \min\{M(fu, f(x_n), t), M(gu, fu, t), M(g(x_n), f(x_n), t)\}.$$

It follows that $m(f, g) \to M(gu, u, t)$, as $n \to \infty$.

Now, by

$$\varphi\left(\frac{1}{M(gu, g(x_n), t)} - 1\right) \leq \varphi\left(\frac{1}{m(f, g)} - 1\right) - \psi\left(\frac{1}{m(f, g)} - 1\right),$$

on taking $n \to \infty$, we get

$$\varphi\left(\frac{1}{M(gu, u, t)} - 1\right) \leq \varphi\left(\frac{1}{M(gu, u, t)} - 1\right) - \psi\left(\frac{1}{M(gu, u, t)} - 1\right),$$

which implies $gu = u$. Therefore $fu = u = gu$ and hence u is a common fixed point of f and g.

For uniqueness of a fixed point, we shall assume that z be another common fixed point of f and g. Then, by putting $x = u$ and $y = z$, we have

$$\begin{aligned} m(f, g) &= \min\{M(fu, fz, t), M(gu, fu, t), M(gz, fz, t)\} \\ &= \min\{M(u, z, t), M(u, u, t), M(z, z, t)\} \\ &= M(u, z, t). \end{aligned}$$

Consequently

$$\varphi\left(\frac{1}{M(gu,gz,t)}-1\right)=\varphi\left(\frac{1}{M(u,z,t)}-1\right)$$

$$\leq\varphi\left(\frac{1}{M(u,z,t)}-1\right)-\psi\left(\frac{1}{M(u,z,t)}-1\right),$$

which implies $M(u,z,t)=1$, that holds if and only if $u=z$. Therefore u is a unique common fixed point of f and g.

Example 4.3.1.3 Let $X=[0,1]$ and $M(x,y,t)=\frac{t}{t+|x-y|}$, for all $x,y\in X, t>0$. Define $\varphi,\psi:[0,\infty)\to[0,\infty)$ by $\varphi(t)=t$ and $\psi(t)=\frac{t}{2}$, for all $t>0$. Define also $f,g:X\to X$ by

$$f(x)=\frac{x}{2} \text{ for all } x\in[0,1] \text{ and } g(x)=\begin{cases}\frac{x}{16}, & \text{if } x\in[0,\frac{1}{2}];\\ 0, & \text{if } x\in(\frac{1}{2},1].\end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $x_n=\frac{1}{2n}$, for all $n\geq 1$. Then, we have

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}g(x_n)=\lim_{n\to\infty}fg(x_n)=\lim_{n\to\infty}gf(x_n)=\lim_{n\to\infty}ff(x_n)=\lim_{n\to\infty}gg(x_n)=0$$

and

$$\lim_{n\to\infty}M(fg(x_n),gf(x_n),t)=1.$$

Therefore, the maps f and g are sub-compatible. It is easy to show that f and g are (f,g)-reciprocally continuous and satisfy the contractive condition (4.3.1). Thus, all the hypotheses of Theorem 4.3.1.2 hold and $x=0$ is a unique common fixed point of f and g.

Remark 4.3.1.4 Theorem 4.3.1.2 extends and complements the related results of ([30]) and references therein.

4.3.2 Cyclic (φ,ψ)-Weak Contaction

In 2010, Pacurar and Rus ([115]) introduced the concept of cyclic ϕ-contraction and proved a fixed point theorem for cyclic ϕ-contraction in complete metric spaces. Later

Gopal, Imdad, Vetro and Hasan([40]) introduced the notion of cyclic weak ϕ-contraction in fuzzy metric spaces. For other results in partial metric spaces one can refer to [28].

Definition 4.3.2.1([115]) Let X be a nonempty set, m be a positive integer and $g : X \to X$ an operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to g if

(i) $X_i,\ i = 1, 2, \ldots, m$ are nonempty sets,

(ii) $g(X_1) \subset X_2, \ldots, g(X_{m-1}) \subset X_m, g(X_m) \subset X_1$.

Example 4.3.2.2([40]) Let $X = \mathbb{R}$. Assume $A_1 = A_3 = [-2, 0]$ and $A_2 = A_4 = [0, 2]$, so that $Y = \bigcup_{i=1}^{4} A_i = [-2, 2]$. Define $g : Y \to Y$ such that $g(x) = -\frac{x}{2}$, for all $x \in Y$. It is clear that $Y = \bigcup_{i=1}^{4} A_i$ is a cyclic representation of Y.

Results in[40] inspired us to introduce the notion of cyclic weak (φ, ψ)-contraction in fuzzy metric spaces.

Definition 4.3.2.3 Let $(X, M, *)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $g : Y \to Y$ is called a cyclic weak (φ, ψ)-contraction if the following conditions hold:

(i) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to g;

(ii) there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ such that

$$\varphi\left(\frac{1}{M(gx, gy, t)} - 1\right) \leq \varphi\left(\frac{1}{M(x, y, t)} - 1\right) - \psi\left(\frac{1}{M(x, y, t)} - 1\right), \quad (4.3.3)$$

for any $x \in A_i,\ y \in A_{i+1}$ $(i = 1, 2, \cdots, m,$ where $A_{m+1} = A_1)$ and each $t > 0$.

Now, we develop the following result.

Theorem 4.3.2.4 Let $(X, M, *)$ be a fuzzy metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ be G-complete. Suppose that there exists a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ. If $g : Y \to Y$ is a continuous cyclic weak (φ, ψ)-contraction, then g has a unique fixed point $u \in \bigcap_{i=1}^{m} A_i$.

Proof Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$ and set $x_n = gx_{n-1}$ $(n \geq 1)$. Clearly, we get $M(x_n, x_{n+1}, t) = M(g(x_{n-1}), g(x_n), t)$ for any $t > 0$. Besides for any $n \geq 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then by (4.3.3), for $t > 0$ we have

$$\begin{aligned} \varphi\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) &= \varphi\left(\frac{1}{M(g(x_{n-1}), g(x_n), t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right) - \psi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right) \\ &\leq \varphi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right). \end{aligned} \tag{4.3.4}$$

It implies that $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$ for all $n \geq 1$ and so $\{M(x_{n-1}, x_n, t)\}$ is a non-decreasing sequence of positive real numbers in $(0, 1]$.

Let $S(t) = \lim_{n\to\infty} M(x_{n-1}, x_n, t)$. Now, we show that $S(t) = 1$ for all $t > 0$. If not, there exists some $t > 0$ such that $S(t) < 1$. Then, on making $n \to \infty$ in (4.3.4), we obtain

$$\varphi\left(\frac{1}{S(t)} - 1\right) \leq \varphi\left(\frac{1}{S(t)} - 1\right) - \psi\left(\frac{1}{S(t)} - 1\right)$$

which is a contradiction. Therefore $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty$.

Now, for each positive integer p, we have

$M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, t/p) * M(x_{n+1}, x_{n+2}, t/p) * \cdots * M(x_{n+p-1}, x_{n+p}, t/p)$. It follows that

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1,$$

and hence $\{x_n\}$ is a G-Cauchy sequence.

Since Y is G-complete, then there exists $z \in Y$ such that $\lim_{n\to\infty} x_n = z$. On the other hand, by the condition (i) of Definition 4.3.2.3, it follows that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = 1, 2, \ldots, m$. As Y is G-complete, from each A_i, $i = 1, 2, \ldots, m$, one can extract a subsequence of $\{x_n\}$ that converges to z. In virtue of the fact that each A_i, $i = 1, 2, \ldots, m$, is closed, we conclude that $z \in \bigcap_{i=1}^{m} A_i$ and so $\bigcap_{i=1}^{m} A_i \neq \emptyset$. Obviously, $\bigcap_{i=1}^{m} A_i$ is closed and G-complete. Now, consider the restriction of g on $\bigcap_{i=1}^{m} A_i$, i.e., $g|\bigcap_{i=1}^{m} A_i : \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i$ which satisfies the assumptions of Theorem (4.3.1.2) and thus, $g|\bigcap_{i=1}^{m} A_i$ has a unique fixed point in $\bigcap_{i=1}^{m} A_i$, say u, which is obtained by iteration from the starting point $x_0 \in Y$. To this aim, we have to show that $x_n \to u$ as $n \to \infty$, where $x_n = gx_{n-1}$ $(n \geq 1)$. We have proved that, for every $x_0 \in X$, the sequence $\{x_n\}$ converges to some $z \in X$. Then, by (4.3.3), we have

$$\varphi\left(\frac{1}{M(x_n,u,t)} - 1\right) \leq \varphi\left(\frac{1}{M(x_{n-1},u,t)} - 1\right) - \psi\left(\frac{1}{M(x_{n-1},u,t)} - 1\right).$$

Now, letting $n \to \infty$, we get

$$\varphi\left(\frac{1}{M(z,u,t)} - 1\right) \leq \varphi\left(\frac{1}{M(z,u,t)} - 1\right) - \psi\left(\frac{1}{M(z,u,t)} - 1\right)$$

which is a contradiction if $M(z, u, t) < 1$, and so, we conclude that $u = z$. Obviously, u is a unique fixed point of g.

Remark 4.3.2.5 Theorem 4.3.2.4 extends and generalizes the related results of ([45, 115, 43]) in fuzzy metric spaces via cyclic weak (φ, ψ)-contraction.

Example 4.3.2.6 Let $X = \mathbb{R}$ and $M(x, y, t) = \frac{t}{t+|x-y|}$, for all $x, y \in X, t > 0$. Assume $A_1 = A_2 = \cdots = A_m = [0, 1]$, so that $Y = \bigcup_{i=1}^{m} A_i = [0, 1]$ and define $g : Y \to Y$ by $g(x) = \frac{x^2}{4}$ for all $x \in Y$. Furthermore, if $\varphi, \psi : [0, \infty) \to [0, \infty)$ are defined by $\psi(s) = \frac{s}{4}$

and $\varphi(s) = \frac{s}{2}$ for all $s \geq 0$, we have

$$\begin{aligned} \varphi\left(\frac{1}{M(gx,gy,t)} - 1\right) &= \frac{|x^2 - y^2|}{8t} \\ &\leq \frac{|x-y|}{4t} = \varphi\left(\frac{1}{M(x,y,t)} - 1\right) - \psi\left(\frac{1}{M(x,y,t)} - 1\right). \end{aligned}$$

Clearly, g is a cyclic weak (φ, ψ)-contraction and all the conditions of Theorem (4.3.2.4) are satisfied. Therefore g has a unique fixed point $0 \in \bigcap_{i=1}^{m} A_i$.

4.4 Section III

Here we have proved common fixed point theorem of *Presič* type mapping in Fuzzy metric spaces which extends the results of George ([129]).

Now we shall introduced the concept of $2k$ - weakly compatible pair of maps. This notion is analogous to the notion introduced by Rao, Kishore and Ali ([92]).

Definition 4.4.1 ([92]) Let X be a non empty set and $T : X^{2k} \to X, f : X \to X.(f, T)$ is said to be $2k$ - weakly compatible pair, if $f(T(p, p, \cdots, p)) = T(fp, fp, \cdots, fp)$ whenever $p \in X$ such that $fp = T(p, p, \cdots, p)$.

4.4.1 Main Results

Consider a function $\phi : [0, 1]^{2k} \to [0, 1]$ such that:

(i) ϕ is an increasing function, i.e.,

$x_1 \leq y_1, x_2 \leq y_2, \cdots, x_{2k} \leq y_{2k}$ implies $\phi(x_1, x_2, \cdots, x_{2k}) \leq \phi(y_1, y_2, \cdots, y_{2k})$.

(ii) $\phi(t, t, t, \cdots) \geq t$, for all $t \in X$.

(iii) ϕ is continuous in all variables.

Now we prove our main theorem:

Theorem 4.4.1.1 Let $(X, M, *)$ be a Fuzzy Metric space, k is a positive integer and $S, T : X^{2k} \to X,\ f : X \to X$ be mappings satisfying

$$M(S(x_1, x_2, \cdots, x_{2k-1}, x_{2k}), T(x_2, x_3, \cdots, x_{2k}, x_{2k+1}), qt) \geq \phi\{M(fx_i, fx_{i+1}, t) : 1 \leq i \leq 2k\} \tag{4.4.1}$$

for all $x_1, x_2, \cdots, x_{2k+1}$ in $X, 0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

$$M(T(y_1, y_2, \cdots, y_{2k-1}, y_{2k}), S(y_2, y_3, \cdots, y_{2k}, y_{2k+1}), qt) \geq \phi\{M(fy_i, fy_{i+1}, t) : 1 \leq i \leq 2k\} \tag{4.4.2}$$

for all $y_1, y_2, \cdots, y_{2k+1}$ in X,

$$M(S(u, u, \cdots, u), T(v, v, \cdots, v), qt) > M(fu, fv, t), \forall u, v \in X \quad \text{with} \quad (u \neq v) \tag{4.4.3}$$

Suppose that $f(X)$ is complete and either (f, S) or (f, T) is $2k$ - weakly compatible pair.

Then there exists a unique point $p \in X$ such that

$$fp = p = S(p, p, \cdots, p) = T(p, p, \cdots, p).$$

Proof Suppose $x_1, x_2, \cdots, x_{2k}$ are arbitrary points in X and for $n \in \mathbb{N}$.

Define $f(x_{2k+2n-1}) = S(x_{2n-1}, x_{2n}, x_{2n+1}, \cdots, x_{2n+2k-2})$ and

$f(x_{2k+2n}) = T(x_{2n}, x_{2n+1}, x_{2n+2}, \cdots, x_{2n+2k-1})$.

Let $\alpha_n = M(f(x_n), f(x_{n+1}), qt)$.

Claim: $\alpha_n \geq (\dfrac{K - \theta^n}{K + \theta^n})^2$ for all $n \in \mathbb{N}$,

where $\theta = \dfrac{1}{2q}$, and $K = min\{\dfrac{\theta(1 + \sqrt{\alpha_1})}{1 - \sqrt{\alpha_1}}, \dfrac{\theta^2(1 + \sqrt{\alpha_2})}{1 - \sqrt{\alpha_2}}, \cdots, \dfrac{\theta^{2k}(1 + \sqrt{\alpha_{2k}})}{1 - \sqrt{\alpha_{2k}}}\}$

By selection of k we have $\alpha_n \geq (\dfrac{K - \theta^n}{K + \theta^n})^2$ for $n = 1, 2, \cdots, 2k$.

Now

$$\alpha_{2k+1} = M(f(x_{2k+1}), f(x_{2k+2}), qt)$$

$$= M(S(x_1, x_2, \cdots, x_{2k-1}, x_{2k}), T(x_2, x_3, \cdots, x_{2k}, x_{2k+1}), qt)$$

$$\geq \phi\{M(f(x_i), f(x_{i+1}), t) : i = 1, 2, \cdots, 2k\} \quad \text{by} \quad (4.4.1)$$

$$= \phi\{\alpha_1, \alpha_2, \cdots, \alpha_{2k-1}, \alpha_{2k}\}$$

$$\geq \phi\{(\frac{K-\theta^1}{K+\theta^1})^2, (\frac{K-\theta^2}{K+\theta^2})^2, \cdots, (\frac{K-\theta^{2k-1}}{K+\theta^{2k-1}})^2, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2\}$$

$$\geq \phi\{(\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, \cdots, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2\}$$

$$\geq (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2$$

$$\geq (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2$$

Thus $\alpha_{2k+1} \geq (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2$

Similarly,

$$\alpha_{2k+2} = M(f(x_{2k+2}), f(x_{2k+3}), qt)$$

$$= M(T(x_2, x_3, \cdots, x_{2k}, x_{2k+1}), S(x_3, x_4, \cdots, x_{2k+2}), qt)$$

$$\geq \phi\{M(f(x_i), f(x_{i+1}), t) : i = 2, 3 \cdots, 2k+1\} \quad \text{by} \quad (4.4.1)$$

$$= \phi\{\alpha_i : i = 2, 3, \cdots 2k+1\}$$

$$\geq \phi\{(\frac{K-\theta^2}{K+\theta^2})^2, (\frac{K-\theta^3}{K+\theta^3})^2, \cdots, (\frac{K-\theta^{2k}}{K+\theta^{2k}})^2, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2\}$$

$$\geq \phi\{(\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2, \cdots, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2, (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2\}$$

$$\geq (\frac{K-\theta^{2k+1}}{K+\theta^{2k+1}})^2$$

$$\geq (\frac{K - \theta^{2k+2}}{K + \theta^{2k+2}})^2$$

Thus $\alpha_{2k+2} \geq (\frac{K - \theta^{2k+2}}{K + \theta^{2k+2}})^2$

Hence the claim is true.

Now, by claim for any $n, p \in \mathbb{N}$, we have

$$\begin{aligned} M(f(x_n), f(x_{n+p}), t) &\geq M(f(x_n), f(x_{n+1}), \frac{t}{2}) * M(f(x_{n+1}), f(x_{n+2}), \frac{t}{2^2}) \\ &* \cdots * M(f(x_{n+p-1}), f(x_{n+p}), \frac{t}{2^p}) \\ &\geq \alpha_n * \alpha_{n+1} * \cdots * \alpha_{n+p-1} \\ &\geq (\frac{K - 2^n}{K + 2^n})^2 * (\frac{K - 2^{2n}}{K + 2^{2n}})^2 * \cdots * (\frac{K - 2^{np}}{K + 2^{np}})^2 \\ &\to 1 * 1 * \cdots * 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{f(x_n)\}$ is Cauchy sequence. Since $f(X)$ is a complete then there will exists z in $f(X)$ such that $\lim_{n\to\infty} f(x_n) = z$. There exists $p \in X$ such that $z = fp$. Then for any integer n, using (4.4.1) and (4.4.2), we have

$$\begin{aligned} M(S(p, p, p \cdots, p), fp, t) &= lim_{n\to\infty} M(S(p, p, \cdots, p), fx_{2n+2k-1}, t) \\ &= \lim_{n\to\infty} M(S(p, p, \cdots, p), S(x_{2n-1}, x_{2n}, \cdots, x_{2n+2k-2}), t) \\ &\geq \lim_{n\to\infty} M(S(p, p, \cdots, p), T(p, p, \cdots, x_{2n-1}), \tfrac{t}{2}) \\ &\quad * M(T(p, p, \cdots, x_{2n-1}), S(p, p, \cdots, x_{2n-1}, x_{2n}), \tfrac{t}{2^3}) * \cdots \\ &\quad * M(T(p, x_{2n-1}, \cdots, x_{2n+2k-3}), S(x_{2n-1}, x_{2n}, \cdots, x_{2n+2k-2}), \tfrac{t}{2^{k-1}}) \\ &\geq \lim_{n\to\infty} \phi\{M(fp, fp, t), M(fp, fp, t), \cdots, M(fp, fx_{2n-1}, t)\} \\ &\quad * \phi\{M(fp, fp, t), M(fp, fp, t), \cdots, M(fx_{2n-1}, fx_{2n}, t)\} \\ &\quad * \cdots * \phi\{M(fp, fx_{2n-1}, t), M(fx_{2n-1}, fx_{2n}, t), \end{aligned}$$

$$\cdots, M(fx_{2n+2k-3}, fx_{2n+2k-2}, t)\} \to 1,$$

i.e., $M(S(p,p,\cdots,p), f(p), t) = 1$ and so $c(f,T) \neq \Phi$ where $c(f,T)$ denote the set of all coincidence points of the mappings f and T.

So that

$$S(p,p,\cdots p) = fp \tag{4.4.4}$$

Consider

$$\begin{aligned} M(f(p), T(p,p,\cdots,p), t) &= M(S(p,p,\cdots,p), T(p,p,\cdots,p), t) \\ &\geq \phi\{M(f(p),f(p),t), M(f(p),f(p),t), \cdots, M(f(p),f(p),t)\} \\ &\geq M(f(p),f(p),t) \\ &= 1 \end{aligned}$$

Thus

$$T(p,p,\cdots,p) = f(p). \tag{4.4.5}$$

Now suppose that (f,S) is $2k$ - weakly compatible pair. Then we, have

$f(S(p,p,\cdots,p)) = S(f(p), f(p), \cdots, f(p))$.

$f^2(p) = f(f(p)) = f(S(p,p,\cdots,p)) = S(f(p),f(p),\cdots,f(p))$.

Suppose that $f(p) \neq p$. Then from (4.4.3), we have

$$\begin{aligned} M(f^2(p), f(p), t) &= M(S(f(p),f(p),\cdots,f(p)), T(p,p,\cdots,p), t) \\ &\geq \{M(f^2(p),f(p),t), M(f^2(p),f(p),t), \cdots, M(f^2(p),f(p),t)\} \\ &\geq M(f^2(p),f(p),t). \end{aligned}$$

It is a contradiction.

Therefore $f(p) = p$. Now from (4.4.4) and (4.4.5), we have

$f(p) = p = S(p,p,\cdots,p) = T(p,p,\cdots,p)$.

Uniqueness of p:

Suppose there exists a point $q \neq p$ in X such that

$f(q) = q = S(q, q, \cdots, q) = T(q, q, \cdots, q)$

Consider

$$M(f(p), f(q), t) = M(S(p, p, \cdots, p), T(q, q, \cdots, q), t)$$

$$\geq \phi\{M(f(p), f(q), t), M(f(p), f(q), t), \cdots, M(f(p), f(q), t)\}$$

$$\geq M(f(p), f(q), t).$$

It is a contradiction. Therefore $p = q$.

When $S = T$ and $2k$ is replaced by k in main theorem, we get the following corollary.

Corollary 4.4.1.2 Let $(X, M, *)$ be a FM - space, k is a positive integer $T : X^k \to X$, $f : X \to X$ be mappings satisfying

$$M(T(x_1, x_2, \cdots, x_k), T(x_2, x_3, \cdots, x_{k+1}, qt) \geq \phi\{M(f(x_i), f(x_{i+1}), t) : 1 \leq i \leq 2k\}$$

for all $x_1, x_2, \cdots, x_{k+1}$ in $X, 0 < q < \frac{1}{2}$ and $t \in [0, \infty)$

$$M(T(u, u, \cdots, u), T(v, v, \cdots, v), qt) > M(f(u), f(v), t), \forall u, v \in X \quad \text{with} \quad u \neq v$$

Suppose that $f(X)$ is complete and (f, T) is k - weakly compatible pair. Then there exists a unique point $p \in X$ such that $f(p) = p = S(p, p, \cdots, p) = T(p, p, \cdots, p)$.

Chapter 5

PERIODIC POINTS IN A COMPLEX VALUED METRIC SPACE

5.1

In this chapter, first we shall introduce the concept of complex valued metric space by modifying the same introduced initially by Azam, Fisher and Khan([4]). As an application of these concepts, we shall construct a theorem and its application in integral equations.

The first result of this type was due to Sehgal([160]) and his result was generalized by Guseman([96]), Khanzanchi([101]), Rhoades and Ray([20]) and Murthy and Pathak([125]).

Complex Valued Metric Space:

Let X be a nonempty set and let $\rho : X \times X \to \mathbb{C}$, where $\mathbb{C}$ is a set of complex numbers in which ordering is not the same as in the set of real numbers. We recall some important definitions, lemmas and theorems for our further study on common fixed points in complex valued metric spaces.

Let $\mathbb{C}$ be a set of complex numbers and $\xi_1, \xi_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ which follows:

The elements $\xi_1, \xi_2 \in \mathbb{C}$ are partially ordered denoted by

$$\xi_1 \preceq \xi_2 \Rightarrow Re(\xi_1) < Re(\xi_2), Im(\xi_1) < Im(\xi_2),$$

or

$$\xi_1 \succeq \xi_2 \Rightarrow Re(\xi_1) < Re(\xi_2), Im(\xi_1) < Im(\xi_2),$$

Two elements $\xi_1, \xi_2 \in \mathbb{C}$, and

$$\xi_1 \preceq \xi_2 \quad (\text{or} \quad \xi_1 \succeq \xi_2)$$

If one of the following conditions holds:

(i) $Re(\xi_1) = Re(\xi_2), Im(\xi_1) = Im(\xi_2)$,

(ii) $Re(\xi_1) < Re(\xi_2), Im(\xi_1) < Im(\xi_2)$,

or $Re(\xi_1) > Re(\xi_2), Im(\xi_1) > Im(\xi_2)$,

(iii) $Re(\xi_1) < Re(\xi_2), Im(\xi_1) = Im(\xi_2)$,

or $Re(\xi_1) > Re(\xi_2), Im(\xi_1) = Im(\xi_2)$,

(iv) $Re(\xi_1) = Re(\xi_2), Im(\xi_1) < Im(\xi_2)$,

or $Re(\xi_1) = Re(\xi_2), Im(\xi_1) > Im(\xi_2)$

In particular,

$\xi_1 \lnapprox \xi_2 \quad (\text{or} \quad \xi_1 \gnapprox \xi_2)$, if $\xi_1 \neq \xi_2$ and one of $(ii), (iii)$ and (iv) is satisfied.

We will also write $\xi_1 \prec \xi_2$(or $\quad \xi_1 \succ \xi_2$), if (ii) is satisfied.

Note that $0 \preceq \xi_1 \lnapprox \xi_2 \Longrightarrow \mid \xi_1 \mid < \mid \xi_2 \mid$,

For all $\xi_1, \quad \xi_2, \quad \xi_3 \in \mathbb{C}$

$$\xi_1 \preceq \xi_2, \xi_2 \prec \xi_3 \Longrightarrow \xi_1 \prec \xi_2.$$

Definition 5.1.1 Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \to \mathbb{C}$, satisfies

(CM_1) $0 \preceq \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0 \quad$ if and only if $\quad x = y$;

(CM_2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(CM_3) $\rho(x, y) \preceq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a complex valued metric on X and (X, ρ) is called a complex valued metric space.

Remark 5.1.2 For various examples on complex valued metric spaces refer ([4], [49], [171]).

A point x in X is called an interior point of a set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : \rho(x, y) \prec r\} \subseteq A.$$

A point x of X is called a limit point of A, if there exists $B(x, r)$ centered at x with radius r which contains at least one point of A other than x. i.e. $(B(x, r) - \{x\}) \cap A_{\sim x} \neq \phi$.

A subset G of X is said to be open if each point of G is an interior point of G. A subset B of X is said to be closed if each limit point of B is in B.

The family $F = \{B(x, r) : x \in X, 0 \prec r\}$ is a sub-basis for the Hausdorff topology on X.

A sequence $\{x_n\}$ of X is said to be a convergent sequence and converges to a point $x \in X$, if for a given $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$, there exists a positive integer n_0 such that $\rho(x_n, x) \prec \epsilon$ for all $n > n_0$.

A sequence $\{x_n\}$ of X is said to be a Cauchy sequence, if for a given $\epsilon \in \mathbb{C}$ with $\epsilon \succ 0$ there exists a positive integer n_0 such that $\rho(x_n, x_m) \prec \epsilon$ for all $m, n > n_0$.

A complex valued metric space (X, ρ) is said to be complete, if every Cauchy sequence in X is a convergent sequence.

Lemma 5.1.3 Let (X, ρ) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\mid \rho(x_n, x) \mid \to 0$ as $n \to \infty$.

Lemma 5.1.4 Let (X, ρ) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\mid \rho(x_n, x_{n+m}) \mid \to 0$ as $m, n \to \infty$.

5.1.1 Main Results

We prove the following theorem in complex valued metric space.

Theorem 5.1.1.1 Let E and F be two self mappings of a complete complex metric space (X, ρ) such that there exist positive integers $p(x)$ and $q(x)$ such that for each $x, y \in X$,

$$\rho(E^{p(x)}x, F^{q(y)}y) \preceq \frac{\alpha(\rho(x,y))\rho(x, E^{p(x)}x)\rho(y, F^{q(y)}y)}{\rho(x,y) + \rho(x, F^{q(y)}y) + \rho(y, E^{p(x)}x)} + \beta(\rho(x,y))\rho(x,y) \tag{5.1.1}$$

where $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$.

Then E and F have a unique common fixed point in X.

Proof Let x_0 be an arbitrary point of X and define the sequence $\{x_n\}$

$$x_n = \begin{cases} E^{p(x_{n-1})}x_{n-1}, & \text{when } n \text{ is odd,} \\ F^{q(x_{n-1})}x_{n-1}, & \text{when } n \text{ is even} \end{cases} \tag{5.1.2}$$

for $n = 1, 2, 3 \cdots$

If $x_{2n+1} = x_{2n+2}$, then $\{x_n\}$ is a Cauchy sequence.

Now suppose that $x_{2n+1} \neq x_{2n+2}$ for each $p(x) \neq q(y)$. Then

$$\rho(x_{2n+1}, x_{2n+2}) = \rho(E^{p(x_{2n})}x_{2n}, F^{q(x_{2n+1})}x_{2n+1})$$

$$\preceq \frac{\alpha(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, E^{p(x_{2n})}x_{2n})\rho(x_{2n+1}, F^{q(x_{2n+1})}x_{2n+1})}{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n}, F^{q(x_{2n+1})}x_{2n+1}) + \rho(x_{2n+1}, E^{p(x_{2n})}x_{2n})}$$

$$+\beta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1})$$

$$\preceq \frac{\alpha(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1})\rho(x_{2n+1}, x_{2n+2})}{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n}, x_{2n+2}) + \rho(x_{2n+1}, x_{2n+1})}$$

$$+\beta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1})$$

$$\preceq \alpha(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}) + \beta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}),$$

since $\rho(x_{2n+1}, x_{2n+2}) \preceq \rho(x_{2n}, x_{2n+2}) + \rho(x_{2n}, x_{2n+1}))$.

Or equivalently

$$\rho(x_{2n+1}, x_{2n+2}) \preceq \delta(\rho(x_{2n}, x_{2n+1}))\rho(x_{2n}, x_{2n+1}), \tag{5.1.3}$$

where $\delta = \alpha + \beta < 1$.

Similarly, replacing x by x_{2n+2} and y by x_{2n+3}, we have

$$\rho(x_{2n+2}, x_{2n+3}) \preceq \alpha(\rho(x_{2n+1}, x_{2n+2})) + \beta(\rho(x_{2n+1}, x_{2n+2}))\rho(x_{2n+1}, x_{2n+2}),$$

or equivalently

$$\rho(x_{2n+2}, x_{2n+3}) \preceq \delta(\rho(x_{2n+1}, x_{2n+2}))\rho(x_{2n+1}, x_{2n+2}) \tag{5.1.4}$$

where $\delta = \alpha + \beta < 1$.

From (5.1.3) and (5.1.4), we have

$$\rho(x_n, x_{n+1}) \preceq \delta(\rho(x_{n-1}, x_n))\rho(x_{n-1}, x_n)$$

for all $n \in N$, which implies that

$$| \rho(x_n, x_{n+1}) | \ \leq \ | \delta(\rho(x_{n-1}, x_n)) || \rho(x_{n-1}, x_n) | \ \leq \ | \rho(x_{n-1}, x_n) | . \qquad (5.1.5)$$

Therefore $\{\rho(x_{n-1}, x_n)\}_{n \in N}$ is monotonically decreasing and bounded below.

Hence $| \rho(x_{n-1}, x_n) | \to d$ for some $d \geq 0$.

To prove that $d = 0$, we shall assume $d > 0$. Taking the limit as $n \to \infty$ in (5.1.5), we have

$$| \delta(\rho(x_{n-1}, x_n)) | \to 1.$$

Since $\delta \in \triangle$, $| \rho(x_{n-1}, x_n) | \to 0$, is a contradiction. Therefore, we have $d = 0$.

Now, we shall show that the sequence $\{x_n\}$ is a Cauchy sequence. It is easy and enough to show that $\{x_{2n}\}$ is a Cauchy sequence.

Since X is complete, every Cauchy sequence in X is convergent and converges to a point u (say) in X.

Suppose $F(u) \neq u$. Then from (5.1.1), we have

$$\begin{aligned} \rho(x_{2n+1}, F^{q(u)}u) &= \rho(E^{p(x_{2n})}x_{2n}, F^{q(u)}u) \\ &\preceq \frac{\alpha(\rho(x_{2n}, u))\rho(x_{2n}, E^{p(x_{2n})}x_{2n})\rho(u, F^{q(u)}u)}{\rho(x_{2n}, u) + \rho(x_{2n}, F^{q(u)}u) + \rho(u, E^{p(x_{2n})}x_{2n})} \\ &\quad +\beta(\rho(x_{2n}, u))\rho(x_{2n}, u). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, it follows that $| \rho(u, F^{q(u)}u) | \to 0$.

Thus $F^{q(u)}u = u$. Similarly we can show that $E^{p(u)}u = u$ and so u is a periodic point of E and F.

Now we shall show that this point is unique. If possible, let v be another periodic point of E and F. i.e. $E^{p(u)}u = u$ and $F^{q(v)}v = v$. Then

$$\rho(u, v) = \rho(E^{p(u)}u, F^{q(v)}v)$$

$$\preceq \frac{\alpha(\rho(u,v))\rho(u,E^{p(u)}u)\rho(v,F^{q(v)}v)}{\rho(u,v)+\rho(u,F^{q(v)}v)+\rho(u,E^{p(u)}u)}$$

$$+\beta(\rho(u,v))\rho(u,v)$$

and so

$$\rho(u,v) \preceq \beta(\rho(u,v))\rho(u,v)$$

which implies that $u = v$, since $0 \leq \beta(\rho(u,v)) < 1$).

Hence u is a unique periodic point of E and F.

Now $Eu = EE^{p(u)}u = E^{p(u)}E(u)$ implies that $E(u)$ is a periodic point of E. From the uniqueness of $u.E(u) = u$. Similarly, $F(u) = u$. Hence, u is a common fixed point of E and F.

This completes the proof.

Theorem 5.1.1.2 Let E and F be two self mappings of a complete complex metric space (X,d) such that there exists positive integers $p(x)$ and $q(x)$ such that for each $x, y \in X$,

$$\rho(E^{p(x)}x, F^{q(y)}y) \preceq \alpha \frac{\rho(x,E^{p(x)}x)\rho(y,F^{q(y)}y)}{\rho(x,y)+\rho(x,F^{q(y)}y)+\rho(y,E^{p(x)}x)} + \beta\rho(x,y) \tag{5.1.6}$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

Then E and F have a unique common fixed point in X.

Proof Let x_0 be an arbitrary point of X and define the sequence $\{x_n\}$

$$x_n = \begin{cases} E^{p(x_{n-1})}x_{n-1}, & \text{when n is odd,} \\ F^{q(x_{n-1})}x_{n-1}, & \text{when n is even} \end{cases} \tag{5.1.7}$$

for $n = 1, 2, 3 \cdots$.

If $x_{2n+1} = x_{2n+2}$, then $\{x_n\}$ is a Cauchy sequence.

If not let $x_{2n+1} \neq x_{2n+2}$ for each $p(x) \neq q(y)$. Then

$$\rho(x_{2n+1}, x_{2n+2}) = \rho(E^{p(x_{2n})}x_{2n}, F^{q(x_{2n+1})}x_{2n+1})$$

$$\preceq \alpha \frac{\rho(x_{2n}, E^{p(x_{2n})}x_{2n})\rho(x_{2n+1}, F^{q(x_{2n+1})}x_{2n+1})}{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n}, F^{q(x_{2n+1})}x_{2n+1}) + \rho(x_{2n+1}, E^{p(x_{2n})}x_{2n})}$$

$$+\beta\rho(x_{2n}, x_{2n+1})$$

$$\preceq \alpha \frac{\rho(x_{2n}, x_{2n+1})\rho(x_{2n+1}, x_{2n+2})}{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n}, x_{2n+2}) + \rho(x_{2n+1}, x_{2n+1})}$$

$$+\beta\rho(x_{2n}, x_{2n+1}),$$

since

$$\rho(x_{2n+1}, x_{2n+2}) \preceq \rho(x_{2n}, x_{2n+2}) + \rho(x_{2n}, x_{2n+1}))$$

and so

$$\rho(x_{2n+1}, x_{2n+2}) \preceq (\alpha + \beta)\rho(x_{2n}, x_{2n+1}).$$

Thus, for any $m > n$,

$$\rho(x_n, x_m) \preceq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m)$$

$$\preceq (\delta^n + \delta^{n+1} + \delta^{n+2} + \cdots + \delta^{m-1})\rho(x_0, x_1)$$

and so

$$\mid \rho(x_n, x_m) \mid \leq \frac{\delta^n}{1-\delta} \mid d(x_0, x_1) \mid \to 0$$

as $m, n \to \infty$, where $\delta = \alpha + \beta < 1$. This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, every Cauchy sequence in X is a convergent and so the sequence $\{x_n\}$ converges to a point u in X.

Suppose $F(u) \neq u$, then from (5.1.6), we have

$$\rho(x_{2n+1}, F^{(u)}u) = \rho(E^{p(x_{2n})}x_{2n}, F^{q(u)}u)$$

$$\preceq \alpha \frac{\rho(x_{2n}, E^{p(x_{2n})}x_{2n})\rho(u, F^{q(u)}u)}{(\rho(x_{2n}, u) + \rho(x_{2n}, F^{q(u)}u) + \rho(u, E^{p(x_{2n})}x_{2n}))} + \beta\rho(x_{2n}, u).$$

Letting $n \to \infty$ in the above inequality implies that $\mid \rho(u, F^{(u)}u) \mid \to 0$. Thus $F^{(u)}u = u$. Similarly we can show that $E^{(u)}u = u$ and so u is a periodic point of E and F.

Now we shall show that this point is unique. Suppose if possible that v is another periodic point of E and F. i.e. $E^{(u)}u = u$ and $F^{(v)}v = v$. Then

$$\begin{aligned}\rho(u, v) &= \rho(E^{p(u)}u, F^{q(v)}v)\\ &\preceq \alpha \frac{\rho(u, E^{p(u)}u)\rho(v, F^{q(v)}v)}{\rho(u, v) + \rho(u, F^{q(v)}v) + \rho(u, E^{p(u)}u)} + \beta\rho(u, v)\\ &= \beta\rho(u, v),\end{aligned}$$

which implies that $u = v$, since $\beta < 1$. Hence u is a unique periodic point of E and F. Now $Eu = EE^{p(u)}u = E^{p(u)}E(u)$ implies that $E(u)$ is a periodic point of E. Then from the uniqueness of u, we have $E(u) = u$. Similarly, $F(u) = u$. Hence, u is a common fixed point of E and F.

This completes the proof of the theorem.

As an immediate consequence of the above theorems, we have the following corollaries:

Corollary 5.1.1.3 Let E be a self mapping of a complete complex metric space (X, ρ) such that there exists positive integers $p(x)$ and $q(x)$ such that for each $x, y \in X$,

$$\rho(E^{p(x)}x, E^{q(y)}y) \preceq \frac{\alpha(\rho(x, y))\rho(x, E^{p(x)}x)\rho(y, E^{q(y)}y)}{\rho(x, y) + \rho(x, E^{q(y)}y) + \rho(y, E^{p(x)}x)} + \beta(\rho(x, y))\rho(x, y)$$

where $\alpha, \beta : \mathbb{C}_+ \to [0, 1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$. Then E has a unique common fixed point in X.

Corollary 5.1.1.4 Let E be a self mappings of a complete complex metric space (X, ρ) such that there exists positive integers $p(x)$ and $q(x)$ such that for each $x, y \in X$,

$$\rho(E^{p(x)}x, E^{q(y)}y) \preceq \alpha \frac{\rho(x, E^{p(x)}x)\rho(y, E^{q(y)}y)}{\rho(x, y) + \rho(x, E^{q(y)}y) + \rho(y, E^{p(x)}x)} + \beta\rho(x, y)$$

where $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$. Then E has a unique common fixed point in X.

Corollary 5.1.1.5 Let E be a self mapping of a complete complex metric space (X, ρ) such that there exists positive integer p such that for each $x, y \in X$,

$$\rho(E^p x, E^p y) \preceq \frac{\alpha(\rho(x,y))\rho(x, E^p x)\rho(y, E^p y)}{\rho(x,y) + \rho(x, E^p y) + \rho(y, E^p x)} + \beta(\rho(x,y))\rho(x,y)$$

where $\alpha, \beta : \mathbb{C}_+ \to [0,1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$. Then E has a unique common fixed point in X.

Corollary 5.1.1.6 Let E be a self mappings of a complete complex metric space (X, ρ) such that there exists positive integer p such that for each $x, y \in X$,

$$\rho(E^p x, E^p y) \preceq \alpha \frac{\rho(x, E^p x)\rho(y, E^p y)}{\rho(x,y) + \rho(x, E^p y) + \rho(y, E^p x)} + \beta \rho(x,y)$$

where $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$. Then E has a unique common fixed point in X.

Corollary 5.1.1.7 Let E be a self mapping of a complete complex metric space (X, ρ) such that for each $x, y \in X$,

$$\rho(Ex, Ey) \preceq \frac{\alpha(\rho(x,y))\rho(x, Ex)\rho(y, Ey)}{\rho(x,y) + \rho(x, Ey) + \rho(y, Ex)} + \beta(\rho(x,y))\rho(x,y)$$

where $\alpha, \beta : \mathbb{C}_+ \to [0,1)$ such that for all $x, y \in X, \alpha(x) + \beta(x) < 1$. Then E has a unique common fixed point in X.

Corollary 5.1.1.8 Let E be a self mappings of a complete complex metric space (X, ρ) such that for each $x, y \in X$,

$$\rho(Ex, Ey) \preceq \alpha \frac{\rho(x, Ex)\rho(y, Ey)}{\rho(x,y) + \rho(x, Ey) + \rho(y, Ex)} + \beta \rho(x,y)$$

where $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$. Then E has a unique common fixed point in X.

5.1.2 Application

As an application of the above result we shall construct the following:

Let $X = C([a,b], R^n)$, $\quad a > 0$ and let $\rho : X \times X \to \mathbb{C}$ be defined by

$$\rho(x,y) = max_{t\in[a,b]} \parallel x(t) - y(t) \parallel_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a}.$$

Consider the Urysohns integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + g(t), \tag{5.1.8}$$

$$x(t) = \int_a^b K_2(t,s,x(s))ds + g(t), \tag{5.1.9}$$

where $t \in [a,b] \subseteq \mathrm{R}, x, g, h \in X$ and $K_1, \quad K_2 : [a,b] \times [a,b] \times \mathrm{R}^n \to \mathrm{R}^n$.

Suppose $K_1, \quad K_1$ are such that $F_x, \quad G_x \in X$ for all $x \in X$, where

$$F_x(t) = \int_a^b K_1(t,s,x(s))ds, \tag{5.1.10}$$

$$G_x(t) = \int_a^b K_2(t,s,x(s))ds \tag{5.1.11}$$

for all $t \in [a,b]$.

If there exist two mappings $\alpha, \quad \beta : C_+ \to [0,1]$ such that for all $x, y \in X$ the following hold:

(i) $\alpha(t) + \beta(t) < 1$;

(ii) the mapping $\gamma : C_+ \to [0,1]$ defined by $\gamma(x)\frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;

(iii) $\| F_x(t) - G_y(t) + g(t) - h(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a}$

$\preceq \alpha(\max_{t\in[a,b]} A(x,y)(t))A(x,y) + \beta(\max_{t\in[a,b]} A(x,y)(t))B(x,y)$,

where

$A(x,y)(t) = \| x(t) - y(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a}$

and

$$B(x,y)(t) = \frac{\| F_x(t) + g(t) - x(t) \|_\infty \quad \| G_y(t) + h(t) - y(t) \|_\infty}{\| F_x(t) + g(t) - y(t) \|_\infty \quad + \quad \| G_y(t) + h(t) - x(t) \|_\infty \quad + \quad d(x,y)}.$$

Then the system of integral equations (5.1.8) and (5.1.9) has a unique common solution.

Proof Define $S, T : X \to X$ by $S(x) = F_x + g$ and $T(x) = G_y + h$. Then

$$\rho(Sx, Ty) = \max_{t\in[a,b]} \| F_x(t) - G_y(t) + g(t) - h(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(x, Sx) = \max_{t\in[a,b]} \| F_x(t) + g(t) - x(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(y, Ty) = \max_{t\in[a,b]} \| G_y(t) + h(t) - y(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(y, Sx) = \max_{t\in[a,b]} \| F_x(t) + g(t) - y(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a},$$

$$\rho(x, Ty) = \max_{t\in[a,b]} \| G_y(t) + h(t) - x(t) \|_\infty \sqrt{1+a^2} \quad e^{i\tan^{-1}a}.$$

Then we can easily see that for $x, y \in X$,

$$\rho(Ex, \quad Fy) \preceq \alpha \frac{\rho(x, \quad Ex)\rho(y, \quad Fy)}{\rho(x, \quad y) \quad + \quad \rho(x, \quad Fy) \quad + \quad \rho(y, \quad Ex)} \quad + \quad \beta \quad \rho(x,y).$$

By applying Theorem 5.1.1.1 we get the solution to (5.1.8) and (5.1.9) of Urysohn's Integral Equations which is unique.

Chapter 6

COMMON FIXED POINTS FOR INTEGRAL TYPE CONTRACTIVE CONDITION IN MENGER SPACES

6.1

In this section, we shall establish a fixed point theorem for integral type contraction condition in the probabilistic metric space for a pair of compatible maps of type (A) (or compatible of type (P) or weak compatible of type (A)) maps.

In 2002, Branciari ([5]) introduced the integral contraction type of condition for obtaining fixed points. His theorem generalizes the Banach Fixed Point Theorem([134]) in metric spaces.

Generalizations of integral type contraction condition for a pair of maps are given by Rhoades ([15]), Kumar ,Chugh and Vats ([153]), etc.

Menger spaces:

First, we recall that a real valued function defined on the set of real numbers is known as a distribution function if it is non-decreasing, left-continuous and

$\inf f(x) = 0, \sup f(x) = 1$.

An example of a distribution function is Heavyside function $\mathsf{H}z(x)$, defined by

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$$H(x) = \begin{cases} 0 & : \text{if } x \leq 0 \\ 1 & : \text{if } x > 0 \end{cases}$$

Definition 6.1.1 A probabilistic metric space (PM-space) is a pair (X, F) where X is a nonempty set and F is a function defined on $X \times X$ into L, the set of distribution functions, such that if x, y and z are points of X, then

(i) $F(x, y; 0) = 0$;

(ii) $F(x, y; t) = H(t)$ if and only if $x = y$;

(iii) $F(x, y; t) = F(y, x; t)$;

(iiii) $F(x, y; s) = 1$ and $F(y, z; t) = 1$ then $F(x, z; s + t) = 1$ for all $x, y, z \in X$ and $s, t > 0$.

For each x and y in X and for each real number $t \to 0, F(x, y; t)$ is to be thought of as the probability that the distance between x and y is less than t. Of course, any metric space (X, d) may be regarded as a PM-space. Indeed, if (X, d) is a metric space, then the distribution function $F(x, y; t)$ defined by

$$F(x, y; t) = H(\tilde{t}d(x, y))$$

induces a PM-space , where d is a usual metric.

In this section,(X, F) is considered as probabilistic metric space with the condition that $\lim_{t \to \infty} F(x, y; t) = 1$ for all x, y in X.

Definition 6.1.2 A t-norm is a 2-place function $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following;

(a) $\triangle(0,0) = 0$,

(b) $\triangle(0,1) = 1$,

(c) $\triangle(a,b) = \triangle(b,a)$,

(d) If $a \leq c, \quad b \leq d$, then $\triangle(a,b) \leq \triangle(c,d)$,

(e) $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$ for all a,b,c in $[0,1]$.

Definition 6.1.3 A Menger PM-space is a triplet $(X, F, \triangle)$ where (X, F) is a PM-space and $\triangle$ is a t-norm with the following condition:

$$F(x, z; s + t) \geq \triangle(F(x, y; s), F(y, z; t)) \quad \text{for all} \quad x, y, z \in X \quad \text{and} \quad s, t > 0.$$

This inequality is known as Menger's triangle inequality.
In 1966, the notion of contraction mapping on PM-space was first introduced by Seghal([164]). Moreover, "every contraction mapping on a complete Menger space has a unique fixed point". For more detail see, ([87], [124], [71], [72], [76], [106], [164]).

Theorem 6.1.4 Let (X, F) be a PM-space and $f : X \to X$ be an arbitrary mapping on X. Then f is called a contraction if there exist $k \in (0,1)$ such that for all x, y in X and $t > 0$, we have $F(f(x), f(y); kt) \geq F(x, y; t)$.
In 1991, Mishra ([147]) introduced the concept of compatible mappings in PM-space.

Theorem 6.1.5 Let f and g be self mappings on a Menger space $(X, F, \triangle)$. Then the mappings f and g are called compatible, if $\lim_{n\to\infty} F(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$, for all $t > 0$.
In 1992, Cho, Murthy and Stojakovic ([175]) introduced the notion of Compatible mappings of type (A) in Menger Spaces which follows:

Theorem 6.1.6 Let f and g be self mappings on a Menger space $(X, F, \triangle)$. The mappings f and g are called compatible maps of type (A) if $\lim_{n\to\infty} F(fg(x_n), gg(x_n), t) = 1$ and $\lim_{n\to\infty} F(fg(x_n), gg(x_n), t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ for all $t > 0$.

Theorem 6.1.7 ([72]) Let f and g be self mappings on a Menger space $(X, F, \triangle)$. The mappings f and g are called weak compatible maps of type (A) if
$\lim_{n\to\infty} F(fg(x_n), gg(x_n), t) = 1$ or $\lim_{n\to\infty} F(fg(x_n), gg(x_n), t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and for all $t > 0$.

Remrk 6.1.8 Now we state some results, which highlight relation among compatible mappings, compatible mappings of type (A),compatible mappings of type (P) and weak compatible mappings of type (A).

(i) Let f and g be compatible (res. compatible mappings of type (P),compatible mappings of type (A) and weak compatible mappings of type (A)) self-mappings of a Menger space (X, F, Δ) . If $f(t) = g(t)$ for some $t \in X$,then $fg(t) = gf(t)$.

(ii) Let $f, g : (X, F, \Delta) \to (X, F, \Delta)$ be continuous mappings .Then f and g are compatible if and only if f and g are compatible mappings of type (A) (or compatible mappings of type (P), weak compatible mappings of type (A)).

(iii) Let (X, F, Δ) be a Menger space. Let $f, g : X \to X$ be compatible mappings of type (A) (compatible mappings of type (P) ,weak compatible mappings of type (A)). If one of f and g is continuous, then f and g are compatible.

(iv) Let $f, g : (X, F, \Delta) \to (X, F, \Delta)$ be compatible mappings of type (A) (compatible mappings of type (P) ,weak compatible mappings of type (A)) and $fx = gx$ for some $x \in X$, then $fg(x) = gg(x) = gf(x) = gg(x)$.

6.2 Variants of compatible maps

Now, we shall prove our theorems in Menger Spaces:

Theorem 6.2.1 Let f and g be compatible self maps of a complete metric space (X, d) satisfying the following conditions:

(6.2.1) $f(X) \subset g(X)$,

(6.2.2) one of the mapping f or g is continuous

(6.2.3) $\int_0^{d(fx,fy)} \phi(t)dt \leq c.\int_0^{d(gx,gy)} \phi(t)dt$, for each $x, y \in X, c \in [0, 1)$.

where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is a Lebesgue-integrable mapping which is summable, non-negative, and such that

(6.2.4) $\int_0^{\epsilon} \phi(p)dp > 0$ for each $\epsilon > 0$.

Then f and g have a unique common fixed point.

Remark 6.2.2 Rhoades pointed out that inequality (6.2.3) should be as $d(fx, fy) < d(gx, gy)$.

Since $c < 1$, $\int_0^{d(fx,fy)} \phi(t)dt < c.\int_0^{d(gx,gy)} \phi(t)dt$ implies $d(fx, fy) \leq d(gx, gy)$ for all $x, y \in X$.

If $d(fx, fy) \geq d(gx, gy)$, for some $x, y \in X$ then by using (6.2.3), we have

$$\int_0^{d(gx,gy)} \phi(t)dt \leq \int_0^{d(fx,fy)} \phi(t)dt < \int_0^{d(gx,gy)} \phi(t)dt$$

which is a contradiction.

Now in this section, we give an analogue of the Theorem 6.2.1 in the probabilistic metric space setting using compatible of type (A) (or compatible of type (P) or weak compatible of type (A)) maps.

Theorem 6.2.3 Let (X, F, Δ) be a complete Menger space. Let f and g be compatible of type (A) (or compatible of type (P) or weak compatible of type (A)) self maps of X satisfying the following conditions:

(6.2.5) $f(X) \subseteq g(X)$,

(6.2.6) one of the mappings f or g is continuous,

(6.2.7) $\int_0^{1-F(fx,fy,ct)} \phi(p)dp \leq \int_0^{1-F(gx,gy,ct)} \phi(p)dp$, for each $x, y \in X, t > 0, c \in [0,1)$,

where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is a Lebesgue-integral mapping which is summable (with finite integral) on each compact subset of $\mathbf{R}^+$, non-negative, and such that for each $\epsilon > 0$,

(6.2.8) $\int_0^{\epsilon} \phi(p)dp > 0$.

Then f and g have a unique common fixed point.

Proof Let $x_0 \in X$, since $f(X) \subseteq g(X)$, choose $x_1 \in X$ such that $g(x_1) = f(x_0)$. In general, choose x_{n+1} such that $y_n = g(x_{n+1}) = f(x_n)$.

For each integer $n \geq 1$, we have from (6.2.7),

$$\begin{aligned}\int_0^{1-F(y_n,y_{n+1},t)} \phi(p)dp &= \int_0^{1-F(f(x_n),f(x_{n+1}),t)} \phi(p)dp \\ &\leq \int_0^{1-F(g(x_n),g(x_{n+1}),\frac{t}{c})} \phi(p)dp \\ &= \int_0^{1-F(f(x_{n-1}),f(x_n),\frac{t}{c})} \phi(p)dp \\ &\leq \int_0^{1-F(g(x_{n-1}),g(x_n),\frac{t}{c^2})} \phi(p)dp.\end{aligned}$$

In this fashion one can obtain

$$\int_0^{1-F(y_n,y_{n+1},t)} \phi(p)dp = \int_0^{1-F(fy_0,y_1,\frac{t}{c^n})} \phi(p)dp.$$

Letting $n \to \infty$ and using Lebesgue Dominated Convergence Theorem and for $c \in [0,1)$, it follows in view of (6.2.8) that $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$. Now we shall show that $\{y_n\}$ is a Cauchy sequence in X.

For each integer $n \geq 1$, from (6.2.7), we have

$$\int_0^{1-F(y_n,y_{n+1},t)}\phi(t)dt = \int_0^{1-F(f(x_n),f(x_{n+1}),t)}\phi(t)dt$$
$$\leq \int_0^{1-F(g(x_n),g(x_{n+1}),\frac{t}{c})}\phi(t)dt$$
$$= \int_0^{1-F(f(x_{n-1}),f(x_n),\frac{t}{c})}\phi(t)dt$$
$$\leq \int_0^{1-F(g(x_{n-1}),g(x_n),\frac{t}{c^2})}\phi(t)dt.$$

Proceeding this way, we obtain after n-th iteration

$\int_0^{1-F(y_n,y_{n+1},t)}\phi(t)dt \leq \int_0^{1-F(y_0,y_1,\frac{t}{c^n})}\phi(t)dt.$

Letting $\lim_{n\to\infty}$ and using Lebesgue Dominated Convergence Theorem and $c \in [0,1)$, it follows that $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$.

Similarly, we have $\lim_{n\to\infty} F(y_{n+1}, y_{n+2}, t) = 1$.

Now for any positive integer p, we will have

$$F(y_n, y_{n+p}, t) \geq F(y_n, y_{n+1}, \tfrac{t}{p}) * \ldots p - times \ldots * F(y_{n+p-1}, y_{n+p}, \tfrac{t}{p})$$

$$\geq F(y_n, y_{n+1}, \tfrac{t}{p}) * \ldots p - times \ldots * F(y_n, y_{n+1}, \tfrac{t}{p})$$

for any positive integer p.

Since $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$, for $t > 0$, it follows that

$\lim_{n\to\infty} F(y_n, y_{n+p}, t) \geq 1 * \ldots * 1 \geq 1$.

Thus $\{y_n\}$ is a Cauchy sequence in X.

But f and g are compatible mappings of type (A) (or compatible of type (P)or weak compatible of type (A)) ,and g is continuous ,therefore, by Remark 6.1.8., we have $\lim_{n\to\infty} fg(x_n) = gz$.

Now from (6.2.7),we have

$$\int_0^{1-F(fg(x_n),f(x_n),t)}\phi(t)dt \leq \int_0^{1-F(gg(x_n),g(x_n),\frac{t}{c})}\phi(t)dt,$$

On letting $\lim_{n\to\infty}$ and using Lebesgue Dominated Convergence Theorem it follows that

$gz = z$. Again from (6.2.7)

$\int_0^{1-F(f(x_n),fz,t)} \phi(t)dt \leq \int_0^{1-F(g(x_n),gz,\frac{t}{c})} \phi(t)dt,$

Taking $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem it follows in view of (6.2.8) that $fz = z$. Hence z is a common fixed point of f and g.

For Uniqueness:

Suppose that $w(\neq z)$ is another fixed point of f and g. From (iii), we have

$$\begin{aligned} \int_0^{1-F(z,w,t)} \phi(t)dt &= \int_0^{1-F(fz,fw,t)} \phi(t)dt \\ &\leq \int_0^{1-F(g(x_n),gw,\frac{t}{c})} \phi(t)dt \\ &= \int_0^{1-F(z,w,\frac{t}{c})} \phi(t)dt \end{aligned}$$

since $c \in [0, 1)$, therefore $z = w$ and so uniqueness follows.

Example 6.2.4 Let $X = [0, 1]$ be equipped with the usual metric space.
Define $F(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for each $t > 0$. Define mappings $f, g : X \to$ by $f(x) = \frac{x}{3}$ and $g(x) = \frac{x}{2}$ for all $x \in X$.
Clearly $f(X) = [0, \frac{1}{3}] \subset gX = [0, \frac{1}{2}]$.
Moreover, ϕ is defined by $\phi(t) = t$ for $t > 0$ is a Lebesgue integral mapping which is summable (with finite integral) on each compact subset of $\mathbf{R}^+$, non-negative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$.
Now
$\int_0^{1-F(fx,fy,ct)} \phi(p)dp = \int_0^{1-F(gx,gy,t)} \phi(p)dp,$
where $1 - F(fx, fy, ct) = \{\frac{d(x,y)}{(3ct+d(x,y))}\}$
and $1 - F(gx, gy, t) = \{\frac{d(x,y)}{(2t+d(x,y))}\}$.

Thus all the conditions of Theorem (6.2.3) are satisfied with $\phi(t) = t$ for $t > 0, \phi(0) = 0$ and $c \in [\frac{2}{3}, 1)$ also 0 is the unique common fixed point of f and g .

Next, we shall prove a theorem for a weakly compatible maps satisfying a contractive condition of integral type as follows:

Theorem 6.2.5 Let $(X, F, \triangle)$ a Menger space. Suppose f and g are weakly compatible self maps of X satisfying $(6.2.5), (6.2.7), (6.2.8)$ and the following condition:

(6.2.9) any one of $f(X)$ or $g(X)$ is complete

Then f and g have a unique common fixed point.

Proof From proof of Theorem (6.2.3), we conclude that $\{y_n\}$ is a Cauchy sequence in X and from (6.2.9) either $f(X)$ or $g(X)$ is complete, for definiteness assume that $g(X)$ is complete. Note that the sequence $\{y_{2n}\}$ is contained in $g(X)$ so has a limit in $g(X)$. Call it z. Let $u \in g^{-1}z$. Then $gu = z$. Now we show that $fu = z$. From (6.2.7), we have

$$\int_0^{1-F(f(x_n),fu,t)} \phi(p)dp = \int_0^{1-F(g(x_n),gu,\frac{t}{c})} \phi(p)dp,$$

for each $x, y \in X, c \in [0, 1)$. Letting $\lim_{n\to\infty}$ and using Lebesgue Dominated Convergence Theorem, it follows in view of (6.2.8) that $fu = z$.

Since f and g are weakly compatible, it follows that $fz = fgu = gfu = gz$.

Now we show that z is a common fixed point of f and g.

From (6.2.7) , we obtain

$$\begin{aligned}\int_0^{1-F(fz,z,t)} \phi(p)dp &= \int_0^{1-F(fz,fu,t)} \phi(p)dp \\ &\leq \int_0^{1-F(gz,gu,\frac{t}{c})} \phi(p)dp \\ &= \int_0^{1-F(fz,z,\frac{t}{c})} \phi(p)dp\end{aligned}$$

which is a contradiction, since $c \in [0, 1)$, this implies $fz = z = gz$ and therefore , z is a common fixed point of f and g.

For Uniqueness:

Suppose that $w(\neq z)$ is also another common fixed points of f and g. Then from (6.2.7) , we have

$$\int_0^{1-F(z,w,t)} \phi(p)dp = \int_0^{1-F(fz,fw,t)} \phi(p)dp$$
$$\leq \int_0^{1-F(gz,gw,\frac{t}{c})} \phi(p)dp$$
$$= \int_0^{1-F(z,w,\frac{t}{c})} \phi(p)dp$$

which implies that $z = w$, and hence uniqueness follows.

Example 6.2.6 Let $X = [3, 22]$ and d be usual metric on X. let $f, g : X \to X$ be defined by

$$f(x) = \begin{cases} 3 & : \text{if } x = 3 \\ 8 & : \text{if } 3 < x \leq 7 \\ 3 & : \text{if } x > 7 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3 & : \text{if } x = 3 \\ 10 & : \text{if } 3 < x \leq 7 \\ \frac{x+2}{3} & : \text{if } x > 7 \end{cases}$$

Define $\psi, \varphi : [0, \infty) \to [0, \infty)$ where $\psi(t) = (t + 1)^{t+1} - 1$ and $\varphi(t) = \psi'(t)$.

Also define

$F(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for each $t > 0$.

Moreover, $fX = \{3\} \cup \{8\}$, $gX = [3, 8] \cup \{10\}$. Hence $fX \subseteq gX$.

To see that f and g are non-compatible maps, consider the sequence

$\{x_n = 7 + \frac{1}{n}, n \geq 1\}$ in X.

Then $\lim_{n\to\infty} f(x_n) = 3, \lim_{n\to\infty} g(x_n) = 3, \lim_{n\to\infty} fg(x_n) = 8$ and $\lim_{n\to\infty} gf(x_n) = 3$. Hence f and g are non-compatible maps.But they are weakly compatible maps since they commute at coincidence point at $x = 3$. Thus f and g satisfy all the conditions of the theorem 6.2.5 and have a unique common fixed point at $x = 3$.

6.3 Universal weakly compatible maps

Now we shall define universal weakly compatible maps in probabilistic metric spaces which follows:

Theorem 6.3.1 A pair of self-mappings (f, g) of a Menger space (X, F, Δ) is said to be Universal weakly compatible maps if there exists a sequence $\{x_n\} \in X$ such that $F(f(x_n), g(x_n), t) = 1$ for some $t \in X$.

Example 6.3.2 Let $X = [0, +\infty)$. Define $S, T : X \to X$ by

$T(x) = \frac{x}{2}$ and $S(x) = \frac{3x}{4}$, for all $x \in X$.

Consider the sequence $\{x_n\} = \frac{1}{n}$.

Clearly $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = 0$.

Then S and T are Universal weakly compatible maps.

Example 6.3.3 Let $X = [\frac{2}{3}, +\infty)$. Define $f, g : X \to X$ by

$g(x) = \frac{x+1}{3}$ and $f(x) = \frac{2x+1}{3}$, for all $x \in X$.

Suppose that f and g are Universal weakly compatible maps.

Then, there exists a sequence $\{x_n\}$ in X satisfying

$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some $z \in X$. Therefore,

$\lim_{n\to\infty} x_n = 3z - 1$ and $\lim_{n\to\infty} x_n = \frac{3z-1}{2}$.Thus, $z = \frac{1}{3}$,

which is a contradiction, since $\frac{1}{3}$ is not contained in X.

Hence f and g are not Universal weakly compatible maps.

Example 6.3.4 Let $X = [0, 1]$ be equipped with the usual metric $d(x, y) = |x - y|$.

Define $F(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for each $t > 0$.

Hence (X, M, Δ) is a menger space. Also define

$$f(x) = \begin{cases} 1 - x & : \text{if } 0 \le x \le \frac{1}{2} \\ \frac{3}{4} & : \text{if } \frac{1}{2} < x \le 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{2} & : \text{if } 0 \le x \le \frac{1}{2} \\ 0 & : \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

Consider the sequence $\{x_n\} = \{\frac{1}{2} - \frac{1}{n}\}$,

we have $\lim_{n\to\infty} f(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} = \lim_{n\to\infty} g(\frac{1}{2})$.

Thus, the pair (f, g) are Universal weakly compatible maps.

Further, f and g are weakly compatible since $x = \frac{1}{2}$ is their unique coincidence point and

$fg(\frac{1}{2}) = f(\frac{1}{2}) = g(\frac{1}{2}) = gf(\frac{1}{2})$.

We further observe that $d(fg(\frac{1}{2} - \frac{1}{n}), gf(\frac{1}{2} - \frac{1}{n})) \neq 0$,

showing that $\lim_{n\to\infty} F(fgx_n, gfx_n, t) \neq 1$,therefore, the pair (f, g) is non compatible.

Example 6.3.5 Let $X = \mathbf{R}^+$ be equipped with the usual metric $d(x, y) = |x - y|$.

Define $F(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for each $t > 0$.

Hence (X, M, Δ) is a menger space. Also define $f, g : X \to X$ by

$fx = 0$, if $0 < x \leq 1$ and $fx = 1$, if $x > 1$ or $x = 0$; and $gx = [x]$, the greatest integer that is less than or equal to x, for all $x \in X$. Consider a sequence

$\{x_n\} = \{1 + \frac{1}{n}\}, n \geq 2$ in $(1, 2)$, then we have $\lim_{n\to\infty} fx_n = 1 = \lim_{n\to\infty} gx_n$. Similarly for the sequence $\{y_n\} = \{1 - \frac{1}{n}\} n \geq 2$ in $(0, 1)$, we have $\lim_{n\to\infty} fy_n = 0 = \lim_{n\to\infty} gy_n$. Thus the pair (f, g) are Universal weakly compatible maps. However, f and g are not weakly compatible as as each $u_1 \in (0, 1)$ and $u_2 \in (1, 2)$ are coincidence points of f and g, where they do not commute. Moreover, they commute at $x = 0, 1, 2...$ but none of these points are coincidence points of f and g. Further, we note that pair (f, g) is non compatible maps.

Now we are ready to prove a theorem for a pair of weakly compatible maps along with the notion of Universal weakly compatible maps.

Theorem 6.3.6 Let (X, F, Δ) be a Menger space. Suppose f and g are weak compatible self-maps of X satisfying $(6.2.5), (6.2.7), (6.2.8)$ and the following:

(6.2.10) f, g are Universal weakly compatible maps

(6.2.11) $f(X)$ or $g(X)$ is a closed subset X.

Then f and g have a coincidence point. Moreover f and g have a unique common fixed point.

Proof. Since f and g be Universal weakly compatible maps,therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. Since either $f(X)$ or $g(X)$ is a closed subspace of X, for definiteness we assume that $g(X)$ is a closed subset of X. Further, note that the sequence $\{y_{2n}\}$ which is contained in $g(X)$, so there is a limit in $g(X)$. Call it be u such that $u = ga$. Therefore, $\lim_{n\to\infty} fx_n = u = ga = \lim_{n\to\infty} gx_n$ for some $a \in X$. This implies $u = ga \in gX$. Now we show that $u = fa = ga$. From (6.2.7) ,we have

$$\int_0^{1-F(a,x_n,ct)} \phi(p)dp = \int_0^{1-F(ga,gx_n,t)} \phi(p)dp$$

Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem and $c \in [0,1)$ it follows in view of (6.2.8) that $F(fa, fa, ct) \leq F(ga, fa, t)$,this implies that $u = ga = fa$. Thus a is the coincidence point of f and g. Since f and g are weakly compatible, therefore, $fu = fga = gfa = gu$.

Now we show that $fu = u$. Now from (6.2.7),we have
$\int_0^{1-F(fu,fa,ct)} \phi(p)dp = \int_0^{1-F(gu,ga,t)} \phi(p)dp$ which in turns implies that $fu = u$. Hence u is the unique common fixed point of f and g.

For Uniqueness:

Suppose that $w(\neq z)$ is also another fixed point of f and g. From (6.2.7), we have
$\int_0^{1-F(z,w,t)} \phi(t)dt = \int_0^{1-F(fz,fw,t)} \phi(t)dt$
$< \int_0^{1-F(gz,gw,\frac{t}{c})} \phi(t)dt = \int_0^{1-F(z,w,\frac{t}{c})} \phi(t)dt$ since $c \in [0,1)$, therefore $z = w$ and so uniqueness follows.

Now we give an illustrate example to discuss the validly of out main theorem 6.3.6.

Example 6.3.7 Let $X = [0,1]$ be equipped with the usual metric space. Define $F(x,y,t) = \frac{t}{t+d(x,y)}$ for all $x,y \in X$ and for each $t > 0$. Hence (X, M, Δ) is a menger space. Also define $f,g : X \to X$ by $f(x) = \frac{x}{3}$, and $g(x) = \frac{x}{2}$, for all $x \in X$.Clearly

$f(X) = [0, \frac{1}{3}] \subset gX = [0, \frac{1}{2}]$.Moreover, ϕ defined by $\phi(t) = t$ for $t > 0$ is a Lebesgue -integral mapping which is summable (with finite integral) on each compact subset of $\mathbf{R}^+$, non-negative, and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \phi(t)dt > 0$$

. Now,

$$\int_0^{1-F(fx,fy,ct)} \phi(p)dp = \int_0^{1-F(gx,gy,t)} \phi(p)dp,$$

where $1 - F(fx, fy, ct) = \{\frac{d(x,y)}{(3ct+d(x,y))}\}$ and $1 - F(gx, gy, t) = \{\frac{d(x,y)}{(2t+d(x,y))}\}$.
Thus all the hypothesis of Theorem(6.3.6) are satisfied with $\phi(t) = t$ for $t > 0$,$\phi(0) = 0$ and $c \in [\frac{2}{3}, 1)$ and 0 is the unique common fixed point of f and g.

6.4 Variants of R-weakly commuting mappings

In 2007, Kohali and Vashistha ([81]) introduced the notions of R-weakly commuting mappings of type (i), R-weakly commuting mappings type (ii) and R-weakly commuting mappings type (iii) in probabilistic metric spaces as follow:
Definition 6.4.1 A pair of self-mappings (f, g) of a Menger space (X, F, Δ) is said to be
(i) R-weakly commuting mappings of type (i) if there exists some $R > 0$ such that $F(gfx, ffx, t) \geq F(fx, gx, \frac{t}{R})$,
(ii) R-weakly commuting mappings of type (ii) if there exists some $R > 0$ such that $F(fgx, ggx, t) \geq F(fx, gx, \frac{t}{R})$,
(iii) R-weakly commuting mappings of type (iii) if there exists some $R > 0$ such that $F(ffx, ggx, t) \geq F(fx, gx, \frac{t}{R})$,for all $x \in X$ and $t > 0$.

Theorem 6.4.2 Theorem 6.2.5 (or Theorem 6.3.6) remains true if weakly compatible property is replaced by any one (retining the rest of hypothesis) of the following:

(a) R-weakly commuting property

(b) R-weakly commuting property of type (i)

(c) R-weakly commuting property of type (ii)

(d) R-weakly commuting property of type (iii)

(e) weakly commuting property.

Proof Since all the conditions of Theorem 6.2.5 (or Theorem 6.3.6) are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (f, g), then using R-weak commutativity one gets $F(fgx, gfx, t) \geq F(fx, gx, \frac{t}{R}) = 1$ which amounts to say that $fgx = gfx$. Thus the pair (f, g) is weakly compatible.

Now applying Theorem 6.2.5 (or Theorem 6.3.6), one concludes that f and g have a unique common fixed point . In case (f, g) is an R-weakly commuting pair of type (ii), then $F(fgx, ggx, t) \geq F(fx, gx, \frac{t}{R}) = 1$ which amounts to say that $fgx = ggx$.

Now $F(fgx, gfx, t) \geq \Delta(F(fgx, ggx, \frac{t}{R}),$

$F(ggx, gfx, \frac{t}{R})) = \Delta(1, 1) = 1$, yielding thereby $fgx = gfx$. Similarly, if pair is R-weakly commuting mappings of type (i) or type (iii) or weakly commuting, then pair (f, g) also commutes at their points of coincidence. Now in view of Theorem 2.3 (or Theorem 3.1) , in all the cases f and g have a unique common fixed point. This completes the proof.

As an application of Theorem 6.2.5 (or Theorem 6.3.6) we prove a common fixed point theorem for two finite families of mappings which runs as follows:

Theorem 6.4.3 Let $\{f_1, f_2 ... f_m\}$ and $\{g_1, g_2, ..., g_n\}$ be two finite families of self-mappings of a Menger space (X, F, Δ) such that $f = f_1 f_2 ... f_m, g = g_1 g_2 ... g_n$, satisfy conditions (6.2.5), (6.2.7) and (6.2.8) and the following:

$g_m(X)$ is a closed subspace of X. Then f and g have a point of coincidence.

Moreover, if $f_if_j = f_jf_i$ and $g_kg_l = g_lg_k$ for all $i, j \in I_1 = \{1, 2, ..., m\}, k, l \in I_2 = \{1, 2, ..., n\}$, then (for all $i \in I_1, k \in I_2$), f_i and g_k have a common fixed point.

Proof The conclusion is immediate i.e., f and g have a point of coincidence as f, and g satisfy all the conditions of Theorem 6.2.5 (or Theorem 6.3.6). Now appealing to component wise commutativity of various pairs, one can immediately prove that $fg = gf$ hence, obviously pairs (f, g) is coincidentally commuting. Note that all the conditions of Theorem 6.2.5 (or Theorem 6.3.6) are satisfied ensuring the existence of a unique common fixed point say z. Now one need to show that z remains the fixed point of all the component maps. For this consider

$f(f_iz) = ((f_1f_2...f_m)f_i)z = (f_1f_2...f_{m-1})((f_mf_i)z) = (f_1...f_{m-1})(f_if_mz)$

$= (f_1...f_{m-2})(f_{m-1}f_i(f_mz)) = (f_1...f_{m-2})(f_if_{m-1}(f_mz))$

$= \ldots = f_1f_i(f_2f_3f_4...f_mz) = f_if_1(f_2f_3...f_mz)$

$= f_i(f_z) = f_iz$. Similarly, one can show that

$f(g_kz) = g_k(fz) = g_kz, f(g_kz) = g_k(gz) = gkz$ and $g(f_iz) = f_i(gz) = f_iz$, which show that (for all I and k) f_iz and g_kz are other fixed points of the pair $(f, g)w$. Now appealing to the uniqueness of common fixed points of both pairs separately, we get $z = f_iz = g_kz$,which shows that z is a common fixed point of f_i, g_k for all i and k.

Corollary 6.4.4 Let f and g be two self-mappings of a Menger space (X, F, Δ) such that f_m and g_n satisfy the conditions $(6.2.5), (6.2.7)$ and $(6.2.8)$.

If one of $f_m(X)$ or $g_n(X)$ is a complete subspace of X, then f and g have a unique common fixed point provided (f, g) commute.

Chapter 7

GENERALIZED (φ, ψ)-WEAK CONTRACTIONS

7.1

In this chapter, we shall deal with such type of contractions involving some control functions. In fact, all contractions are the generalizations of Banach by including the distance function within the real valued function. We have proved some fixed point theorems in G-metric space by using (ϕ, ψ) - contraction condition.

Generalized Metric Spaces:

Now we recall the main definitions for the concept of G-metric space introduce initially by Mustafa and Sims([176]):

Definition 7.1.1 Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(i) $G(x, y, z) = 0$ if and only if $x = y = z$,

(ii) $0 < G(x, y, z)$ for all $x, y \in X$, with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables), and

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

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Then G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 7.1.2 A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 7.1.3 Let (X, G) be a G-metric space. We say that $\{x_n\}$ is:

(vi) a G-Cauchy sequence if, for any $\epsilon > 0$, there is an $n_0 \in N$ (the set of all positive integers) such that for all $n, m, l \geq n_0, G(x_n, x_m, x_l) < \epsilon$;

(vii) a G-convergent sequence if, for any $\epsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0, G(x, x_n, x_m) < \epsilon$.

A G-metric space X is said to be complete if every G-Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in (X, G)$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Definition 7.1.4 Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

We have introduced the notion of f-g reciprocal continuity in G-metric spaces which follows:

Definition 7.1.5 Two self mappings f and g of a G - metric space (X, G) are said to be f-g reciprocally continuous if and only if $\lim_{n\to\infty} ff(x_n) = fu$ and $\lim_{n\to\infty} gg(x_n) = gu$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$.

In 1996 Pathak, Cho, Chang and Lee ([70]) defined a mapping called compatible mapping of type (P) in metric spaces. We redefine the same in the setting of G-metric spaces in the following way:

Definition 7.1.6 Let $f, g : (X, G) \to (X, G)$ be mappings. f and g are said to be compatible maps of type(P) if

$$\lim_{n\to\infty} G(ff(x_n), gg(x_n), gg(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $f(x_n), g(x_n) \to z$, for some $z \in X$, as $n \to \infty$.

Observation 7.1.7 Let f and g be f-g reciprocal continuous mapping of a G - metric spaces (X, G). Then f and g have coincidence point if and only if they are compatible mapping of type(P).

Proof

"If part"Suppose that f and g have coincidence point. Since f and g are f-g reciprocal continuous mappings then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ and $\lim_{n\to\infty} ff(x_n) = fz, \lim_{n\to\infty} gg(x_n) = gz$ for some $z \in X$.
Now by triangle inequality of the G-metric space we have

$$G(ff(x_n), gg(x_n), gg(x_n)) \leq G(ff(x_n), fz, fz) + G(fz, gz, gz) + G(gz, gg(x_n), gg(x_n))$$

. Taking $n \to \infty$, then we have

$$\lim_{n\to\infty} G(ff(x_n), gg(x_n), gg(x_n)) \leq \lim_{n\to\infty} G(ff(x_n), fz, fz) + \lim_{n\to\infty} G(fz, gz, gz)$$

$$+ \lim_{n\to\infty} G(gz, gg(x_n), gg(x_n))$$

which implies that $G(ff(x_n), gg(x_n), gg(x_n)) \to 0$ as $n \to \infty$. Therefore, f and g are compatible maps of type(P).

"Only if part"Now assume that f and g are compatible maps of type(P). Since f and g are f-g reciprocal continuous mappings then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some $z \in X$. But then triangle inequality implies

$$G(fz, gz, gz) \leq G(fz, ff(x_n), ff(x_n)) + G(ff(x_n), gg(x_n), gg(x_n)) + G(gg(x_n), gz, gz)$$

Taking $n \to \infty$ since f and g are f-g reciprocal continuous maps, so we have

$$\lim_{n\to\infty} G(fz, gz, gz) \leq \lim_{n\to\infty} G(fz, ff(x_n), ff(x_n)) + \lim_{n\to\infty} G(ff(x_n), gg(x_n), gg(x_n))$$

$$+ \lim_{n\to\infty} G(gg(x_n), gz, gz)$$

which implies that $G(fz, gz, gz) \to 0$ as $n \to \infty$. Therefore, f and g have coincidence point. This completed the proof.

Now we introduce the concept of compatible mapping in the setting of G - metric spaces which was earlier introduced by Jungck([53]:

Definition 7.1.8 Let f and g be self-maps of a G-metric space (X, G). The maps f and g are said to be compatible if $\lim_{n\to\infty} G(fg(x_n), gf(x_n), gf(x_n), t) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and for all $t > 0$.

Very recently, Bouhadjera and Thobie ([63]) introduced the concept of sub-compatible maps as a generalization of compatible maps in metric spaces. Now, we introduce the notion of sub-compatible mapping in the setting of G-metric spaces.

Definition 7.1.9 Let f and g be self-maps of a G - metric space (X, G). The maps f and g are said to be sub-compatible if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u$ for some $u \in X$ and which satisfy

$$\lim_{n\to\infty} G(fg(x_n), gf(x_n), gf(x_n)) = 0,$$

7.1.1 Main Results

Now we are in a position to prove our main theorem in G-metric space:

Definition 7.1.1.1 Let (X, G) be a G-metric space and $f, g : X \to X$ be given maps. The map g is called a generalized (φ, ψ)-weak contraction with respect to f if there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ such that

$$\varphi(G(gx, gy, gy)) \leq \varphi(m(f, g)) - \psi(m(f, g)) \tag{7.1.1}$$

holds for every $x, y \in X$ and each $t > 0$ with

$$m(f, g) = \max\{G(fx, fy, fy), G(gx, fx, fx), G(gy, fy, fy)\}.$$

If $f = I_X$, where I_X is the identity map, then g is called a generalized (φ, ψ)-weak contraction.

Theorem 7.1.1.2 Let $f, g : X \to X$ be f-g reciprocally continuous self-maps of a complete G - metric space (X, G) such that

1. $g(X) \subseteq f(X)$,
2. g is a generalized (φ, ψ)-weak contraction with respect to f.

If f and g are sub-compatible maps and ψ is a continuous function, then f and g have a unique common fixed point in X, that is, there exists $u \in X$ such that $u = fu = gu$.

Proof Let x_0 be an arbitrary point in X. Since $g(X) \subseteq f(X)$, we can define, for each $n \geq 0$, a sequence of points $x_0, x_1, x_2, \ldots, x_n$ such that x_{n+1} is in the pre-image under

f of $\{gx_n\}$, i. e., $g(x_0) = f(x_1), g(x_1) = f(x_2), \ldots, g(x_n) = f(x_{n+1}), \ldots$. Moreover, we can define a sequence y_n in X by

$$y_n = g(x_n) = f(x_{n+1}). \tag{7.1.2}$$

Suppose that $y_n = y_{n+1}$ for some n. Then by condition (7.1.1) we have easily $y_{n+1} = y_{n+2}$ and so $y_m = y_n$ for every $m > n$. Thus the sequence $\{y_n\}$ is Cauchy. Assume that $y_{n+1} \neq y_n$, for all n. Then, for $x = x_{n+1}$ and $y = x_n$, we have

$$m(f,g) = \max\{G(f(x_{n+1}), f(x_n), f(x_n)), G(g(x_{n+1}), f(x_{n+1}), f(x_{n+1})), G(g(x_n), f(x_n), f(x_n))\}$$

$$= \max\{G(y_n, y_{n-1}, y_{n-1}), G(y_{n+1}, y_n, y_n), G(y_n, y_{n-1}, y_{n-1})\}.$$

So, if $m(f,g) = G(y_{n+1}, y_n, y_n)$, we obtain

$$\varphi(G(g(x_{n+1}), g(x_n), g(x_n)) = \varphi(G(y_{n+1}, y_n, y_n))$$

$$\leq \varphi(G(y_{n+1}, y_n, y_n)) - \psi(G(y_{n+1}, y_n, y_n))$$

which implies that $G(y_{n+1}, y_n, y_n) = 0$, a contradiction as $y_{n+1} \neq y_n$ for all n. Then, we must have $m(f,g) = G(y_n, y_{n-1}, y_{n-1})$ and hence

$$\varphi(G(y_{n+1}, y_n, y_n)) \leq \varphi(G(y_{n-1}, y_n, y_n)) - \psi(G(y_{n-1}, y_n, y_n))$$

$$< \varphi(G(y_n, y_{n-1}, y_{n-1})).$$

Consequently, considering that the φ function is non-decreasing, we have that

$$G(y_n, y_{n+1}, y_{n+1}) > G(y_{n-1}, y_n, y_n), \text{ for all } n, \tag{7.1.3}$$

and hence $\{G(y_{n-1}, y_n, y_n)\}$ is an non - decreasing sequence of positive real numbers in $(0, 1]$. Let $S(t) = lim_{n\to\infty} G(y_{n-1}, y_n, y_n)$. Then we show that $S(t) = 0$. If not, there exists $S(t) < 1$ and $S(t) > 1$. Then from the above inequality, on taking $n \to \infty$, we obtain

$$\varphi S(t) \leq \varphi S(t) - \psi S(t)$$

a contradiction. Therefore $G(y_n, y_{n+1}, y_{n+1}) \to 0$ as $n \to \infty$.

Now, for each positive integer p, we write

$G(y_n, y_{n+p}, y_{n+p}) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G(y_{n+p-1}, y_{n+p}, y_{n+p})$.

It follows that

$$\lim_{n\to\infty} G(y_n, y_{n+p}, y_{n+p}) \leq 0 + 0 + \cdots + 0 = 0,$$

and hence $\{y_n\}$ is a G-cauchy sequence.

Since (X, G) is complete G - metric space, then there exists $u \in X$ such that $y_n \to u$ as $n \to \infty$.

Clearly,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_{n+1}) = u.$$

Now, f-g reciprocal continuity of f and g implies that $ff(x_n) \to fu$ and $gg(x_n) \to gu$. We have $ff(x_n) \to fu$, then by construction of the sequence (7.1.2) we have $fg(x_n) = ff(x_{n+1}) \to fu$. On the other hand, sub-compatibility of f and g yields $\lim_{n\to\infty} G(fg(x_n), gf(x_n), gf(x_n)) = 0$, which implies that $gf(x_n) \to fu$. Now, taking $x = u$ and $y = f(x_n)$, we get

$$m(f, g) = \max\{G(fu, ff(x_n), ff(x_n)), G(gu, fu, fu), G(gf(x_n), ff(x_n), ff(x_n))\}$$

$$\to G(gu, fu, fu), \quad \text{as} \quad n \to \infty.$$

Now, by

$$\varphi(G(gu, gf(x_n), gf(x_n))) \leq \varphi(G(gu, gf(x_n), gf(x_n)) - \psi(G(gu, gf(x_n), gf(x_n))),$$

on taking $n \to \infty$, we get $G(gu, fu, fu) = 0$, which implies that $gu = fu$. It means that u is a coincidence point of f and g.

Now, we shall show that u is also a common fixed point of f and g. For this we assert that $gu = u$, and so $gu = u = fu$. On the other hand, if $gu \neq u$, then taking $x = u$ and $y = x_n$, we have

$$m(f, g) = \max\{G(fu, f(x_n), f(x_n)), G(gu, fu, fu), G(g(x_n), f(x_n), f(x_n))\}.$$

It follows that $m(f, g) \to G(gu, u, u)$, as $n \to \infty$.

Now, by

$$\varphi(G(gu, g(x_n), g(x_n)) \leq \varphi(m(f, g)) - \psi(m(f, g)),$$

on taking $n \to \infty$, we get

$$\varphi(G(gu, u, u)) \leq \varphi(G(gu, u, u)) - \psi(G(gu, u, u)),$$

which implies $gu = u$. Therefore $fu = u = gu$ and hence u is a common fixed point of f and g.

Finally, to prove uniqueness of the fixed point, we suppose that z is another common fixed point of f and g. Then, taking $x = u$ and $y = z$, we have

$$m(f, g) = \max\{G(fu, fz, fz), G(gu, fu, fu), G(gz, fz, fz)\}$$

$$= \max\{G(u, z, z), G(u, u, u), G(z, z, z)\}$$

$$= G(u, z, z).$$

Consequently

$$\varphi(G(gu, gz, gz)) = \varphi(G(u, z, z))$$

$$\leq \varphi(G(u, z, z)) - \psi(G(u, z, z)),$$

which implies $G(u, z, z) = 0$, that holds if and only if $u = z$. Therefore u is a unique common fixed point of f and g.

Example 7.1.1.3 Let $X = [0, 1]$ and (X, G) be the complete G-metric space defined by $G(x, y, z) =| x - y | + | y - z | + | z - x |$, for all $x, y, z \in X$.

Define $\varphi, \psi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = t$, and $\psi(t) = \frac{t}{2}$, for all $t > 0$.

Define also $f, g : X \to X$ by

$$f(x) = \frac{x}{2} \quad \text{for} \quad \text{all} \quad x \in [0,1] \text{ and } g(x) = \begin{cases} \frac{x}{16} & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}.$$

Let $\{x_n\}$ be a sequence in X such that $\{x_n\} = \{\frac{1}{2n}\}, n \in N$. Then we have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = lim_{n\to\infty} fg(x_n) = \lim_{n\to\infty} gf(x_n) = \lim_{n\to\infty} ff(x_n) = \lim_{n\to\infty} gg(x_n) = 0$, so $\lim_{n\to\infty} G(fg(x_n), gf(x_n), gf(x_n)) = 0$.

Hence pair of maps $\{f, g\}$ is sub - compatible and f-g reciprocal continuous. Also we can easily show that the maps f and g satisfies the contractive condition (7.1.1). Thus this example satisfies all the conditions of the above theorem, and $x = 0$ is unique common fixed point for the mappings f and g.

Remark 7.1.1.4 Theorem 7.1.1.1 extends and complements the related results of ([65]) and references therein.

7.1.2 Cyclic (φ, ψ)-Weak Contraction

Since contraction mapping is always continuous so it suggests in a natural way that the question of obtaining fixed points in metric spaces where the self-mapping in a metric spaces not necessarily continuous. The answer of this question given by Kirk, Srinivasan and Veeramani ([168]) and turned the direction of obtaining fixed point by introducing cyclic representations and cyclic contractions. In 2010, Pacurar and Rus ([115]) introduced the concept of cyclic ϕ-contraction and proved a fixed point theorem for cyclic $\phi-$contraction in complete metric spaces. Later Gopal, Imdad, Vetro and Hasan([40]) introduced the notion of cyclic weak $\phi-$contraction in fuzzy metric spaces.

Definition 7.1.2.1 ([115]) Let X be a nonempty set, m a positive integer and $g : X \to$

X an operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to g if

(i) $X_i,\ i = 1, 2, \ldots, m$ are nonempty sets,

(ii) $g(X_1) \subset X_2, \ldots, g(X_{m-1}) \subset X_m, g(X_m) \subset X_1$.

Example 7.1.2.2 ([40]) Let $X = \mathbb{R}$. Assume $A_1 = A_3 = [-2, 0]$ and $A_2 = A_4 = [0, 2]$, so that $Y = \bigcup_{i=1}^{4} A_i = [-2, 2]$. Define $g : Y \to Y$ such that $g(x) = -\frac{x}{2}$, for all $x \in Y$. It is clear that $Y = \bigcup_{i=1}^{4} A_i$ is a cyclic representation of Y.
By drawing the idea of Gopal, Imdad, Vetro and Hasan([40]), we introduce the notion of cyclic weak (φ, ψ)-contraction in the setting of G - metric spaces.

Definition 7.1.2.3 Let (X, G) be a G - metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $g : Y \to Y$ is called a cyclic weak (φ, ψ)-contraction if the following conditions hold:

(i) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to g;

(ii) there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ such that

$$\varphi(G(gx, gy, gy)) \leq \varphi(G(x, y, y)) - \psi(G(x, y, y)), \tag{7.1.4}$$

for any $x \in A_i,\ y \in A_{i+1}$ $(i = 1, 2, \cdots, m,$ where $A_{m+1} = A_1)$ and each $t > 0$.

Now, we are ready to prove the following theorem.

Theorem 7.1.2.4 Let (X, G) be a G - metric space, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ be G-complete. Suppose that there exists a function $\psi : [0, \infty) \to$

$[0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ and an altering distance function φ. If $g : Y \to Y$ is a continuous cyclic weak (φ, ψ)-contraction, then g has a unique fixed point $u \in \bigcap_{i=1}^{m} A_i$.

Proof. Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$ and set $x_n = g(x_{n-1})$ $(n \geq 1)$. Clearly, we get $G(x_n, x_{n+1}, x_{n+1}) = G(g(x_{n-1}), g(x_n), g(x_n))$ Besides for any $n \geq 0$, there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then by (7.1.4), we have

$$\begin{aligned} \varphi(G(x_n, x_{n+1}, x_{n+1})) &= \varphi(G(g(x_{n-1}), g(x_n), g(x_n))) \\ &\leq \varphi(G(x_{n-1}, x_n, x_n)) - \psi(G(x_{n-1}, x_n, x_n)) \\ &\leq \varphi(G(x_{n-1}, x_n, x_n) \end{aligned}$$

which implies that $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$ for all $n \geq 1$ and so $\{G(x_{n-1}, x_n, x_n)\}$ is a non-decreasing sequence of positive real numbers in $(0, 1]$.

Let $S(t) = \lim_{n \to +\infty} G(x_{n-1}, x_n, x_n)$. Now, we show that $S(t) = 0$. If not, there exists $S(t) < 1 \quad \text{and} \quad S(t) > 1$. Then, on making $n \to +\infty$ in (7.1.5), we obtain

$$\varphi(S(t)) \leq \varphi(S(t)) - \psi(S(t))$$

which is a contradiction. Therefore $G(x_n, x_{n+1}, x_{n+1}) \to 1$ as $n \to +\infty$.

Now, for each positive integer p, we have

$$G(x_n, x_{n+p}, x_{n+p}) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{n+p-1}, x_{n+p}, x_{n+p}).$$

It follows that

$$\lim_{n \to +\infty} G(x_n, x_{n+p}, x_{n+p}) \leq 0 + 0 + \cdots 0 = 0,$$

and hence $\{x_n\}$ is a G-Cauchy sequence.

As Y is G-complete, then there exists $z \in Y$ such that $\lim\limits_{n\to+\infty} x_n = z$. On the other hand, by the condition (i) of definition (7.1.2.3), it follows that the iterative sequence $\{x_n\}$ has an infinite number of terms in A_i for each $i = 1, 2, \ldots, m$. Since Y is G-complete, from each A_i, $i = 1, 2, \ldots, m$, one can extract a subsequence of $\{x_n\}$ that converges to z. In virtue of the fact that each A_i, $i = 1, 2, \ldots, m$ is closed, we conclude that $z \in \bigcap\limits_{i=1}^{m} A_i$ and thus $\bigcap\limits_{i=1}^{m} A_i \neq \emptyset$. Obviously, $\bigcap\limits_{i=1}^{m} A_i$ is closed and G-complete. Now, consider the restriction of g on $\bigcap\limits_{i=1}^{m} A_i$, i.e., $g|\bigcap\limits_{i=1}^{m} A_i : \bigcap\limits_{i=1}^{m} A_i \to \bigcap\limits_{i=1}^{m} A_i$ which satisfies the assumptions of Theorem(7.1.1.1) and thus, $g|\bigcap\limits_{i=1}^{m} A_i$ has a unique fixed point in $\bigcap\limits_{i=1}^{m} A_i$, say q, which is obtained by iteration from the starting point $x_0 \in Y$. To this aim, we have to show that $x_n \to u$ as $n \to \infty$. Then, by (7.1.4), we have

$$\varphi(G(x_n, u, u)) \leq \varphi(G(x_{n-1}, u, u)) - \psi(G(x_{n-1}, u, u)).$$

Now, letting $n \to +\infty$, we get

$$\varphi(G(z, u, u)) \leq \varphi(G(z, u, u)) - \psi(G(z, u, u))$$

which is a contradiction if $G(z, u, u) < 0$, and so, we conclude that $u = z$. Obviously, u is a unique fixed point of g.

Example 7.1.2.5 Let $X = \mathbb{R}$ and $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Assume $A_1 = A_2 = \cdots = A_m = [0, 1]$, so that $Y = \bigcup\limits_{i=1}^{m} A_i = [0, 1]$ and define $g : Y \to Y$ by $g(x) = \frac{x^2}{4}$. Furthermore, if $\varphi, \psi : [0, \infty) \to [0, \infty)$ are defined by $\psi(s) = \frac{s}{4}$ and $\varphi(s) = \frac{s}{2}$ for all $s \geq 0$, we have

$$\begin{aligned} \varphi(G(gx, gy, gy)) &= \frac{|x^2 - y^2|}{8} + \frac{|y^2 - x^2|}{8} = \frac{|x^2 - y^2|}{4} \\ &\leq \frac{|x - y|}{4} = \varphi(G(x, y, y))) - \psi(G(x, y, y)). \end{aligned}$$

Clearly, g is a cyclic weak (φ, ψ)-contraction and all the conditions of Theorem (7.1.2.4) are satisfied. Therefore g has a unique fixed point $0 \in \bigcap_{i=1}^{m} A_i$.

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